NUWC-NPT Technical Report 10,839
2 March 1995

Limiting Detection Performance for Random Signals of Unknown Location, Structure, Extent, and Strength

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PREFACE

The work described in this report was sponsored by the Independent Research (IR) Program of the Naval Undersea Warfare Center (NUWC), Division Newport, as Project B10007, entitled **Near-Optimum Detection of Random Signals with Unknown Locations, Structure, Extent, and Strengths.** The IR program is funded by the Office of Naval Research; the NUWC Division Newport program manager is Dr. Kenneth M. Lima (Code 102). The principal investigator is Dr. Albert H. Nuttall (Code 302). Also, the research presented in this report was sponsored by the Science and Technology Directorate of the Office of Naval Research, T. G. Goldsberry (ONR 321W).

The technical reviewer for this report was Dr. Douglas A. Abraham (Code 2121).

Reviewed and Approved: 2 March 1995

Michael Donald W. Counsellor

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Director, Surface Antisubmarine Warfare

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
2 March 1995

3. REPORT TYPE AND DATES COVERED
Progress

4. TITLE AND SUBTITLE
Limiting Detection Performance for Random Signals of Unknown Location, Structure, Extent, and Strength

5. FUNDING NUMBERS
PE 0601152N

6. AUTHOR(S)
Albert H. Nuttall

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Undersea Warfare Center Detachment
39 Smith Street
New London, Connecticut 06320-5594

8. PERFORMING ORGANIZATION REPORT NUMBER
NUWC-NPT TR 10,839

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Office of Naval Research
800 North Quincy Street, BCT 1
Arlington, VA 22217-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

A signal (if present) is located somewhere in a band of frequencies characterized by a total of N search bins. The signal occupies an arbitrary set of $\underline{M}$ of these bins, where not only is $\underline{M}$ unknown, but, also, the locations of the particular $\underline{M}$ occupied bins are unknown. In addition, the (common) signal strength per bin, $\underline{S}$, is unknown.

The limiting detection capability of any processor in this environment has been determined quantitatively by a new bounding procedure that employs an optimum banded processor. The performance levels attained by various practical power-law processors are found to lie within 0.1 dB of the ultimate level, for any value of $\underline{M}$, provided the correct power-law is employed. The best single compromise processor is the 2.4 power-law device, which loses less than 1.2 dB, regardless of the value of $\underline{M}$.

14. SUBJECT TERMS
Detection
Power-Law Processor
Unknown Structure
Likelihood Ratio
Banded Processor
Unknown Extent

15. NUMBER OF PAGES
118

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	SAR

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K_p	number of possible bin patterns, (7)
LR	likelihood ratio, (8)
$\underline{w}$	weight, (9)
X_k	linear sum of data over set L_k, (9)
v	threshold, (10)
e_n	$\exp(\underline{w}\ x_n)$, (11)
S	hypothesized signal power
a	hypothesized strength parameter $1/(1+S)$, (13)
Σ	sum of all data, (14)
a	estimated parameter value, (16)
g(x)	nonlinear transformation, (19)
GLR	generalized likelihood ratio, (21)
K_c	number of possibilities for contiguous band, (35)
I	integer $N/\underline{M}$
OPT-B	optimum banded processor
OPT-Z	optimum processor with zero location information
P_d	detection probability, figure 1
P_f	false alarm probability, figure 1
ν	value of power in power-law processor
PLP	power-law processor, table 1
MGLR	modified generalized likelihood ratio
x_o	breakpoint for MGLR processor
h(x)	nonlinear transformation, (53)
b	breakpoint for device h(x), (53)
MLR	modified likelihood ratio, table 3
FFT	fast Fourier transform

LIMITING DETECTION PERFORMANCE FOR RANDOM SIGNALS OF UNKNOWN LOCATION, STRUCTURE, EXTENT, AND STRENGTH

INTRODUCTION

This technical report is the fourth in a series of NUWC technical reports by this author, covering the following topics:

(a) modified generalized likelihood ratio processors,

(b) generalized likelihood ratio processors,

(c) power-law processors, and

(d) optimum processing,

respectively. Topic (a) was analyzed in [1], resulting in a substantial compilation of receiver operating characteristics for the breakpoint modification considered there. Topic (b) was addressed in [2], again resulting in numerous receiver operating characteristics that quantify the performance of the modification called the sum-of-M-largest processor. Topic (c) was studied in [3], resulting in a recommendation for a device with power-law 2.5, which performed remarkably well, regardless of the (unknown) number of bins occupied by signal and the (unknown but common) signal level.

The overall goal of the extended investigation is to determine classes of processors that perform at or near the optimum level of performance, and which can be easily realized and analyzed, even in these situations of scant knowledge about the detailed signal characteristics. The reader should be familiar with the earlier material and methods before undertaking the current analyses and results.

PROBLEM DEFINITION

The search space consists of N (frequency) bins, each containing independent identically distributed noises of unit power under hypothesis H_0, signal absent. This situation is presumed to be accomplished by an earlier normalization procedure. The number N is under our control and is always a known quantity. When signal is absent, the probability density function of each of the bin output noises is completely known.

When signal is present, hypothesis H_1, the quantity $\underline{M}$ is the actual number of bins occupied by the signal; this is frequently an unknown parameter. The quantity $\underline{L}$ is the actual set of bins occupied by signal, when signal is present; for example, if $\underline{M} = 4$, then we might have for the occupied set, $\underline{L} = \{2,3,7,29\}$, meaning that bins 2,3,7,29 have signal in them. This quantity $\underline{L}$ is always unknown in our investigation. Finally, the quantities $\{\underline{S}_m\}$ are the actual average signal powers in the m-th bin in occupied set $\underline{L}$, when signal is present; these average signal powers are unknown. We shall presume here that all the actual signal powers per bin are equal to a common (unknown) value $\underline{S}$ in the occupied set of bins, $\underline{L}$, and zero elsewhere.

Observe that the actual signal power per bin, $\underline{S}$, can also be interpreted as the actual signal-to-noise power ratio per bin, since the noise power per bin has been normalized at unity.

JOINT PROBABILITY DENSITY FUNCTIONS OF THE SET OF OBSERVATIONS

The joint probability density function governing the complete observation $\{x_n\}$ under hypothesis H_0 follows from (1) and the statistical independence of $\{x_n\}$ as

$$p_0(u_1,\ldots,u_N) = \prod_{n=1}^{N}\{\exp(-u_n)\} \ . \tag{4}$$

Under hypothesis H_1, the signal can land in any set of $\underline{M}$ disjoint bins, out of the total of N search bins. This results in a total number of possibilities $K = (N|\underline{M})$, where the quantity in parentheses is the binomial coefficient. Without any apriori information about signal location, it is presumed that each set occurs with equal probability 1/K. Thus, there are K possible occupancy sets, each of size $\underline{M}$, namely sets $\{L_k\}$ for $1 \le k \le K$. The joint probability density function governing the complete observation $\{x_n\}$ is therefore

$$p_1(u_1,\ldots,u_N) = \sum_{k=1}^{K} \left[\frac{1}{K} \prod_{n\varepsilon L_k}\{\underline{a} \exp(-\underline{a}u_n)\} \prod_{n\not\varepsilon L_k}\{\exp(-u_n)\}\right] , \tag{5}$$

where we used (1) and (2).

SIGNALS WITH PARTIAL LOCATION INFORMATION

In this section, the total number of bins, $\underline{M}$, occupied by a signal (when present) and the (equal) average signal powers per bin, $\underline{S}$, are presumed known to the optimum processor. However, the locations of the $\underline{M}$ occupied bins, in the total of N search bins, are only partially known to the optimum processor. Instead of the $\underline{M}$ occupied bins being allowed to occur anywhere in the range of N bins (which would mean a number of possibilities equal to binomial coefficient $K = (N|\underline{M})$), we presume in this section that a lesser number of possibilities, K_p, for the occupied bin patterns can occur. Further, each of these K_p allowed patterns occurs with equal probability $1/K_p$. In particular, each location pattern L_k, $1 \le k \le K_p$, is a set of size $\underline{M}$, with a structure yet to be specified. For example, if $\underline{M} = 4$, we might have $L_7 = \{5,6,13,19\}$, meaning that, out of the total of N search bins, the particular bins 5,6,13,19 are occupied by signal, for pattern L_7.

$$\sum_{k=1}^{K_p} \exp\left(\underline{w}\ X_k\right) \gtrless v\ . \tag{10}$$

The likelihood ratio test indicates that we must compute __all__ the K_p possible allowed linear sums $\{X_k\}$ in (9), weight each sum by common value $\underline{w}$, exponentiate, and sum over all the possible K_p sets. However, K_p here may be smaller than the earlier value $K = (N|\underline{M})$ encountered in (5), due to fewer allowed set patterns here.

There has been no restriction on the form of the K_p sets $\{L_k\}$ that describe the allowed occupancy patterns for the $\underline{M}$ occupied bins, except that each set L_k is equally likely and is of size $\underline{M}$. If these K_p sets are carefully chosen, the difficult and lengthy calculation prescribed in (9) - (10) for the optimum processor can be greatly simplified, leading to a workable procedure capable of simulation in a reasonable amount of time, for some combinations of parameter values.

If we define random variables

$$e_n = \exp(\underline{w}\ x_n) \quad \text{for } 1 \le n \le N\ , \tag{11}$$

the likelihood ratio test in (10) can be expressed as

$$\sum_{k=1}^{K_p} \prod_{n \varepsilon L_k} e_n \gtrless v\ . \tag{12}$$

Each of the K_p products is of size $\underline{M}$. This form will prove useful in later manipulations, where we will manage to convert

GENERALIZED LIKELIHOOD RATIO PROCESSOR

In this subsection, we presume that $\underline{M}$, the number of occupied signal bins (when present) is known, but that the actual average signal strength $\underline{S}$ and the particular occupied bin locations $\underline{L}$ are not known. Therefore, we hypothesize average signal strength $S \geq 0$ and occupied set L_k, where we let $1 \leq k \leq K_p$, as in the previous subsection. The size of every set L_k is $\underline{M}$. Letting $a = 1/(1+S) \leq 1$, the governing joint probability density function under hypothesis H_1 is, for the hypothesized S and L_k,

$$p_1(u_1,\ldots,u_N) = \prod_{n \in L_k} \{a \exp(-au_n)\} \prod_{n \notin L_k} \{\exp(-u_n)\} \; . \tag{13}$$

For observation $\{x_n\}$, define the random variables

$$\Sigma \equiv \sum_{n=1}^{N} x_n \; , \quad X_k \equiv \sum_{n \in L_k} x_n \quad \text{for } 1 \leq k \leq K_p \; . \tag{14}$$

Then, the joint density in (13) takes on value

$$p_1 \equiv p_1(x_1,\ldots,x_N) = a^{\underline{M}} \exp(-aX_k) \exp[-(\Sigma - X_k)] =$$

$$= \exp(-\Sigma)\, a^{\underline{M}} \exp[(1-a)X_k] \; , \tag{15}$$

which depends on a and k. Holding hypothesized value k fixed, the choice of continuous parameter a that maximizes p_1 in (15) is random variable

$$a = \begin{cases} \underline{M}/X_k & \text{if } X_k > \underline{M} \\ 1 & \text{if } X_k \leq \underline{M} \end{cases} \; , \tag{16}$$

Using the monotonicity of functions exp and g in (19), the generalized likelihood ratio test is therefore simply

$$\max_{k} X_k = \max\left(X_1, \ldots, X_{K_p}\right) \gtrless v \ . \tag{22}$$

Physically, this test for signal presence makes a great deal of sense. First, each variable X_k in (14) is obtained by performing as much linear summation over the observation $\{x_n\}$ as possible, namely over the particular $\underline{M}$ random variables that constitute each set L_k, thereby taking advantage of the known limited signal structure information. Then, the largest sum, out of the total of K_p possibilities, is compared with a threshold for a decision about signal presence or absence. In this generalized likelihood ratio test (22), there is no need to know the actual average signal power $\underline{S}$ or the actual locations $\underline{L}$ of the occupied bins. However, $\underline{M}$ must be known and, along with the partial signal location information $\{L_k\}$, is used to form $\{X_k\}$ in (14).

This test in (22) is not identical to the optimum likelihood ratio test (10) or (12). However, with careful selection of K_p and the attendant allowed sets $\{L_k\}$, the generalized likelihood ratio test in (22) is often easy to realize in practice, and can be simulated in a reasonable amount of time in order to accurately ascertain its performance, whereas optimum test (12) frequently cannot. Also, the optimum test (12) requires knowledge of the actual average signal power per bin, $\underline{S}$.

SIGNALS WITH ZERO LOCATION INFORMATION

Here, the number of bins, $\underline{M}$, occupied by signal and the (equal) average signal powers per bin, $\underline{S}$, are presumed known to the optimum processor, but the locations of the occupied bins are completely unknown. The signal can land in any set of $\underline{M}$ disjoint bins, each set occurring with equal probability 1/K, where K is the binomial coefficient $(N|\underline{M})$. There are now K occupancy sets $\{L_k\}$, each of size $\underline{M}$.

OPTIMUM PROCESSOR

Under hypothesis H_0, the governing probability density function is

$$p_0(u_1,\ldots,u_N) = \prod_{n=1}^{N}\{\exp(-u_n)\} . \tag{24}$$

Under hypothesis H_1, letting $\underline{a} = 1/(1 + \underline{S})$, we have, instead, density

$$p_1(u_1,\ldots,u_N) = \sum_{k=1}^{K} \left[\frac{1}{K} \prod_{n\varepsilon L_k}\{\underline{a}\ \exp(-\underline{a}u_n)\} \prod_{n\not\varepsilon L_k}\{\exp(-u_n)\}\right] . \tag{25}$$

The likelihood ratio for observation $\{x_n\}$ is

$$LR = \frac{p_1(x_1,\ldots,x_N)}{p_0(x_1,\ldots,x_N)} = \frac{1}{K}\ \underline{a}^{\underline{M}} \sum_{k=1}^{K} \prod_{n\varepsilon L_k}\{\exp(x_n[1 - \underline{a}])\} =$$

GENERALIZED LIKELIHOOD RATIO PROCESSOR

In the case where $\underline{M}$ is known, but average signal power $\underline{S}$ is not, we can appeal directly to results (22) and (14) for the generalized likelihood ratio test and interpret the parameters differently. Namely, we obtain the test

$$\max\left(X_1, \ldots, X_K\right) \gtrless v , \tag{31}$$

where

$$X_k = \sum_{n \varepsilon L_k} x_n \quad \text{for } 1 \le k \le K = \binom{N}{\underline{M}} . \tag{32}$$

Notice that K is now the very large integer, $(N|\underline{M})$, required when there is no information on the signal location.

On the other hand, if both $\underline{M}$ and $\underline{S}$ are unknown, we resort to (23) and (14), which gives us the generalized likelihood ratio test

$$\max_{M_1 \le M \le M_2} \left(M \; g\left(\max_k X_k/M\right)\right) \gtrless v , \tag{33}$$

but now where

$$X_k = \sum_{n \varepsilon L_k} x_n \quad \text{for } 1 \le k \le K(M) = \binom{N}{M} . \tag{34}$$

Here, $K(M) = (N|M)$ is a function of the <u>hypothesized</u> number, M, of occupied bins.

SIGNALS WITH SLIDING LOCATION INFORMATION

Here, we again presume $\underline{M}$ and $\underline{S}$ are known, but now we restrict the occupied bin structure to be $\underline{M}$ contiguous bins of unknown (sliding) location. There are now only

$$K_c = N - \underline{M} + 1 \tag{35}$$

possible locations for the contiguous band, which are assumed to be equally probable, apriori. Let the k-th set now be

$$L_k = \{k, k+1, \ldots, k+\underline{M}-1\} \quad \text{for } 1 \le k \le K_c \; . \tag{36}$$

Physically, this corresponds to a signal of known bandwidth, but of unknown center frequency.

likelihood ratio test (where $\underline{S}$ is no longer needed)

$$\sum_{k=1}^{K_c} X_k^2 \gtrless v \,, \tag{41}$$

with X_k given by the sliding sum (39) of $\underline{M}$ linear terms. This approximate likelihood ratio test (41) linearly accumulates the data $\{x_n\}$ as much as possible, to yield the partial sums $\{X_k\}$ in (39), but then adds up quadratic versions of these latter sums. Implementation of test (41) is a relatively simple task, because K_c is generally a manageable number here. A worthwhile alternative is

$$\sum_{k=1}^{K_c} X_k^{\mu} \gtrless v \,, \tag{42}$$

but where the best choice of power μ is not obvious.

It should be noted that the approximate likelihood ratio tests (41) and (42) use a combination of all the partial sums $\{X_k\}$ in making their decision about signal presence or absence; however, the power-law operation accents the larger partial sums at the expense of the smaller ones. Also, tests (41) and (42) require very little computation and can be implemented without knowledge of average signal power level $\underline{S}$; of course, the number $\underline{M}$ of occupied bins must be known in order to evaluate (39). Simulation comparisons of (40) - (42) would have to be conducted in order to ascertain the exact degradations in performance caused by adoption of the suboptimum processors.

OPTIMUM BANDED PROCESSOR

We will now apply the general result for the likelihood ratio test in (10) and (12) to the particular example where the allowed K_p patterns for the $\underline{M}$ occupied bins are formed from a banded signal structure in frequency space. In particular, we divide the total search space of N bins into $\underline{M}$ disjoint bands, each containing $I = N/\underline{M}$ bins. It is presumed that I is an integer.

We now consider a signal, structured so that one, and only one signal, can lie in each band of I bins. This does not correspond to the practical physical situation, where the $\underline{M}$ signals can lie anywhere; rather, this signal model is concocted solely for purposes of simplifying likelihood ratio test (12). It is recognized that the performance of the corresponding optimum banded processor (OPT-B) will be better than the optimum processor with zero location information (OPT-Z), namely (28). Therefore, the performance of the OPT-B will furnish a bound on performance for any practical processor; hopefully, it will also turn out to be a useful and tight bound.

By restricting the $\underline{M}$ signal components to lie in disjoint equal-size bands, and then giving this information to the OPT-B, some partial location information has been furnished. This information is in addition to the values of $\underline{M}$ and $\underline{S}$, which are also presumed known to the OPT-B. However, to compensate as much as possible for this additional information, we require each individual signal component to be equally likely to occupy any of the $I = N/\underline{M}$ bins in each band. This maximizes the confusion for optimally. Since optimum is defined in terms of maximum P_d for a specified P_f, better performance is reflected by a curve being nearer the top of the figure. On the other hand, the bottommost curve for the best power-law device (best ν) must always lie below the OPT-Z, because the power-law device has no knowledge of $\underline{M}$ or $\underline{S}$, nor is it optimum in any sense.

We will demonstrate, by means of several numerical examples, that the top and bottom curves in figure 1 lie virtually on top of each other. That is, the best power-law device performs so well, and the upper bound furnished by the OPT-B is so tight, that the region between these two is a very narrow ribbon. In fact, it will be shown that if the signal power $\underline{S}$, for the best power-law device, is increased by 0.1 dB, then these two corresponding receiver operating characteristics overlap. Thus, we will have succeeded in pinching the unknown receiver operating characteristic of the OPT-Z within a very tight ribbon, and will have effectively determined its performance.

We will also have accomplished two other worthwhile goals. First, we will have found a practical device, namely the power-law processor, which performs virtually at the optimum level of performance in this environment. Second, we will have established such a tight upper bound on performance (by means of the OPT-B) that there is no further need to search for tighter bounds or for the exact performance of the OPT-Z. A discrepancy of 0.1 dB is so small that further significant effort on tighter bounds is hard to justify.

where $e_n = \exp(\underline{w}\, x_n)$, $\underline{w} = \underline{S}/(1 + \underline{S})$. Notice that the sum of $K_p = 16$ products in (47) has been condensed into a single product of sums in (48). This collapsing feature occurs as a direct result of the banded signal structure. Inspection of the final form in (48) reveals that it allows for interaction of every component e_n, $1 \le n \le 4$, in the first band, with every component e_n, $5 \le n \le 8$, in the second band, but no interaction within either band. This is the mathematical representation of the physical separation of signals into disjoint bands.

This collapsing behavior of (12) holds in general for the banded signal structure. Namely, the likelihood ratio test in (12) can be generally expressed exactly in this special case as

$$e_1\, e_{I+1} \cdots e_{N-I+1} + \cdots + e_I\, e_{2I} \cdots e_N =$$

$$= (e_1 + \cdots + e_I)(e_{I+1} + \cdots + e_{2I}) \cdots (e_{N-I+1} + \cdots + e_N) = \tag{49}$$

$$= \prod_{m=1}^{\underline{M}} \sum_{j=1}^{I} e_{mI-I+j} = \prod_{m=1}^{\underline{M}} \sum_{j=1}^{I} \exp\left(\underline{w}\, x_{mI-I+j}\right) \gtrless v\ , \tag{50}$$

where there are I terms in each of the $\underline{M}$ sums. Observe that expansion of (49) would yield $I^{\underline{M}} = K_p$ terms, which is just the number of terms in form (12). However, compact form (49) requires only the evaluation of N exponentials $\{e_n\}$ and $\underline{M}-1$ products, whereas form (12) requires N exponentials and $K_p\,(\underline{M}-1)$ products, a number which is intolerably large when N is large. Result (10) for the likelihood ratio test is not a useful alternative because it requires K_p exponentials and summations.

Furthermore, whereas we are unaware of any method to cut down on the number of calculations, K, required for the OPT-Z likelihood ratio test (28), the OPT-B likelihood ratio test (12) can be reduced to form (49), which is very easy to simulate in a reasonable amount of computer time.

As another example, for $N = 1024$, $\underline{M} = 4$, there follows $K = 4.6E10$ and $K_p = 4.3E9$. Now, the number of alternatives has been reduced by a factor of 10, and the adequacy of the tightness of the performance bound for the OPT-B becomes even more questionable. On the other hand, there are still $K_p = 4.3E9$ alternatives left for the banded signal structure, which indicates that a great deal of uncertainty still remains for the OPT-B to consider.

Finally, for $N = 1024$ and $\underline{M} = 512$, we find $K = 4.5E306$ and $K_p = 1.3E154$. Here, the number of alternatives K_p is approximately the square root of K; that is, K_p is reduced by a factor of 10^{152} relative to K. On the other hand, there still remains a tremendous number of alternatives for the OPT-B to consider, namely on the order of 10^{154}, which may be sufficient to confuse the OPT-B enough to yield a tight performance bound. This question can only be resolved through numerical investigation of the receiver operating characteristics. (For $N = \underline{M} = 1024$, we have $K = K_p = 1$, and the OPT-B and OPT-Z processors become identical, namely the energy detector.)

All the above results hold for the present case of equal signal powers $\underline{S}$ per bin. However, when the signal powers per bin

RECEIVER OPERATING CHARACTERISTICS FOR OPTIMUM BANDED PROCESSOR

We now determine the performance of the optimum banded processor, as characterized by (49) or (50) in general, for any N and $\underline{M}$ (with N/$\underline{M}$ integer). The weight $\underline{w}$ in likelihood ratio test (50) is given by $\underline{w} = \underline{S}/(1 + \underline{S})$, which depends on the particular signal power case under investigation. This means that, in simulating (50), the false alarm probability P_f (as well as the detection probability P_d) must be rerun for each different value of signal power $\underline{S}$ of interest. (This is distinct from the power-law processor, for example, where one curve of P_f could be plotted versus numerous P_d curves; also, P_f was available analytically for some power-law devices, which circumvented the need for simulating P_f at all in those cases.)

This additional simulation expense for the optimum banded processor leads us to adopt the following procedure. We first refer to the performance of the best power-law device [3] to determine what signal power $\underline{S}_p$ was necessary there in order to operate in the neighborhood of the low-quality operating point $P_f = 10^{-3}$ and $P_d = 0.5$. Then, we choose a pair of signal powers that bracket $\underline{S}_p$; that is, we take $\underline{S}_a \le \underline{S}_p \le \underline{S}_b$ and we simulate (50) for both $\underline{S}_a$ as well as $\underline{S}_b$. This gives us a pair of receiver operating characteristics for the optimum banded processor, namely two curves of P_d versus P_f, from which we can interpolate in order to estimate the signal power needed to operate at the low-quality operating point. This entire procedure was repeated for the high-quality operating point $P_f = 10^{-6}$ and $P_d = 0.9$.

A table of the signal powers $\underline{S}$ in decibels required for the optimum banded processor has been extracted from these figures; it is presented under the column labeled OPT-B in table 1 below. Also listed are the corresponding signal powers required for the best power-law processor, under the heading BEST PLP; the last column gives the actual best value of power ν to use. These latter values come from [3; page 28, table 1]. The largest discrepancy in table 1 between OPT-B and BEST PLP is 0.17 dB; however, the coarseness of the search that we conducted in ν values (namely 1, 2, 2.5, 3, ∞) leads us to believe that the largest discrepancy would be closer to 0.1 dB if intermediate values of ν were investigated. Thus, we claim that the best power-law processor is within approximately 0.1 dB of the absolute limit of performance in this environment.

Table 1. Required $\underline{S}$(dB) for $N = 1024$, $P_f = 10^{-3}$, $P_d = 0.5$

$\underline{M}$	OPT-B	BEST PLP	ν
1	12.75	12.77	∞
2	10.03	10.11	∞
4	7.88	8.05	3
8	6.04	6.2	3
16	4.35	4.4	3
32	2.62	2.7	2.5
64	0.71	0.75	2.5
128	-1.54	-1.4	2
256	-4.16	-4.0	2
512	-7.03	-7.00	1
1024	-10.01	-10.01	1

The largest discrepancy in table 2 between OPT-B and BEST PLP is 0.3 dB; however, the coarseness of the search in ν and the difficulty of simulating extremely large trial runs for small probabilities in the 10^{-6} range makes it difficult to accurately estimate the difference in performance between the optimum banded processor and the best power-law processor. A reasonable estimate appears to be about 0.2 dB.

The difficulty of obtaining accurate receiver operating characteristics for very small false alarm probability values is due to the extremely large number of trials required. As an alternative, an analytical approach for determining the false alarm probability for the optimum banded processor is outlined in appendix A. However, it requires two very accurate numerical integrations, coupled with two fast Fourier transforms of rather large size. This procedure was not employed here.

In figure 22, the <u>total</u> signal power, $\underline{M}\ \underline{S}$ in decibels, required to achieve the low-quality operating point, is plotted versus $\underline{M}$ for the power-law processors ν = 1, 2, 2.5, 3, ∞, when N = 1024. This result comes from [3; page 29]. The OPT-B absolute bound in table 1 above is superposed as black dots on this figure. As already anticipated, these results indicate that there is always some power-law processor that performs very close to optimum.

A similar result for the high-quality operating point is given in figure 23; the source of these results is [3; page 30] and the OPT-B absolute bound in table 2 above. The danger of

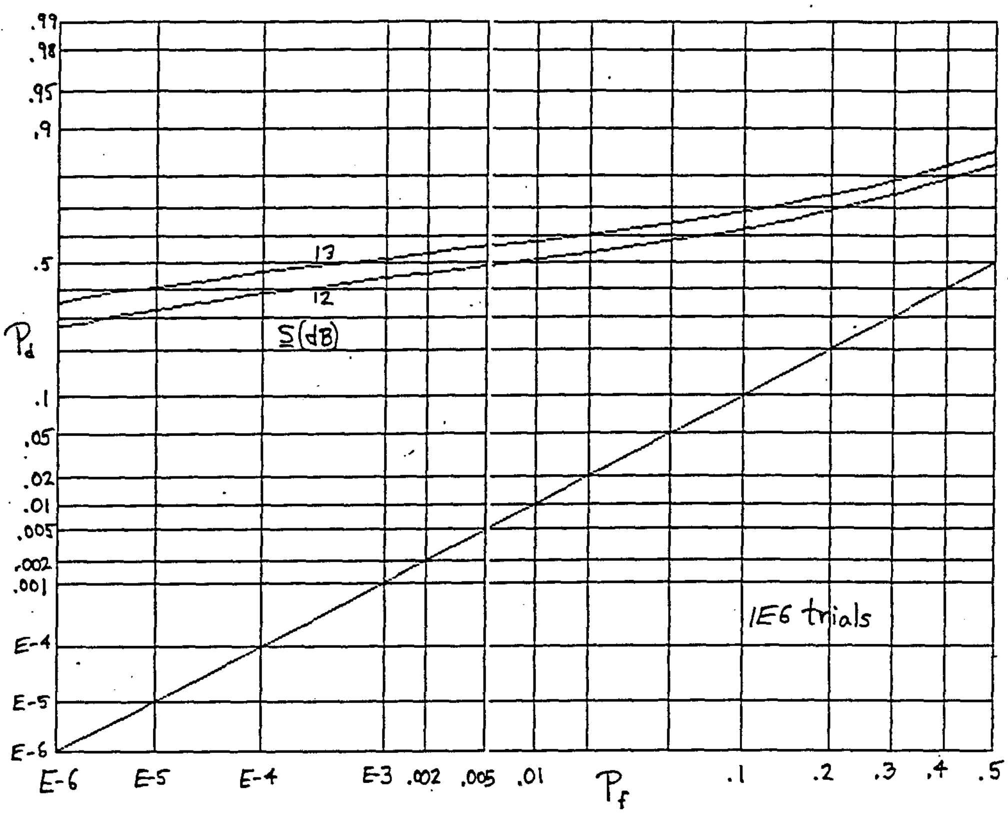

Figure 2. ROC for OPT-B, M = 1, N = 1024, $P_d \sim 0.5$

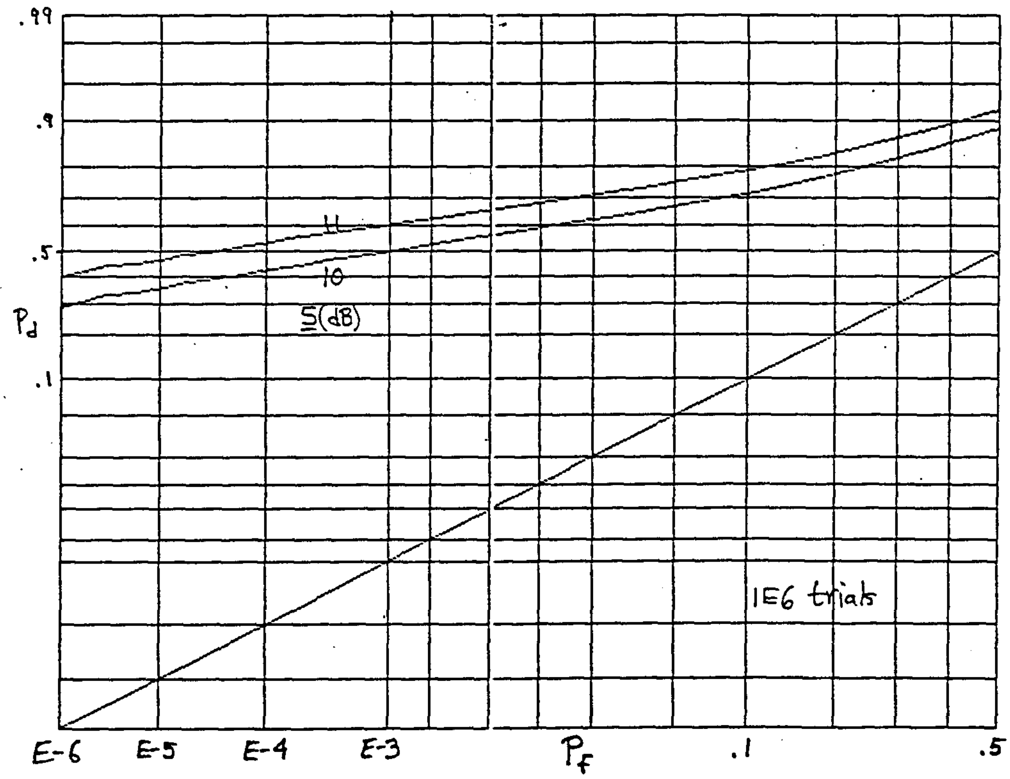

Figure 3. ROC for OPT-B, M = 2, N = 1024, $P_d \sim 0.5$

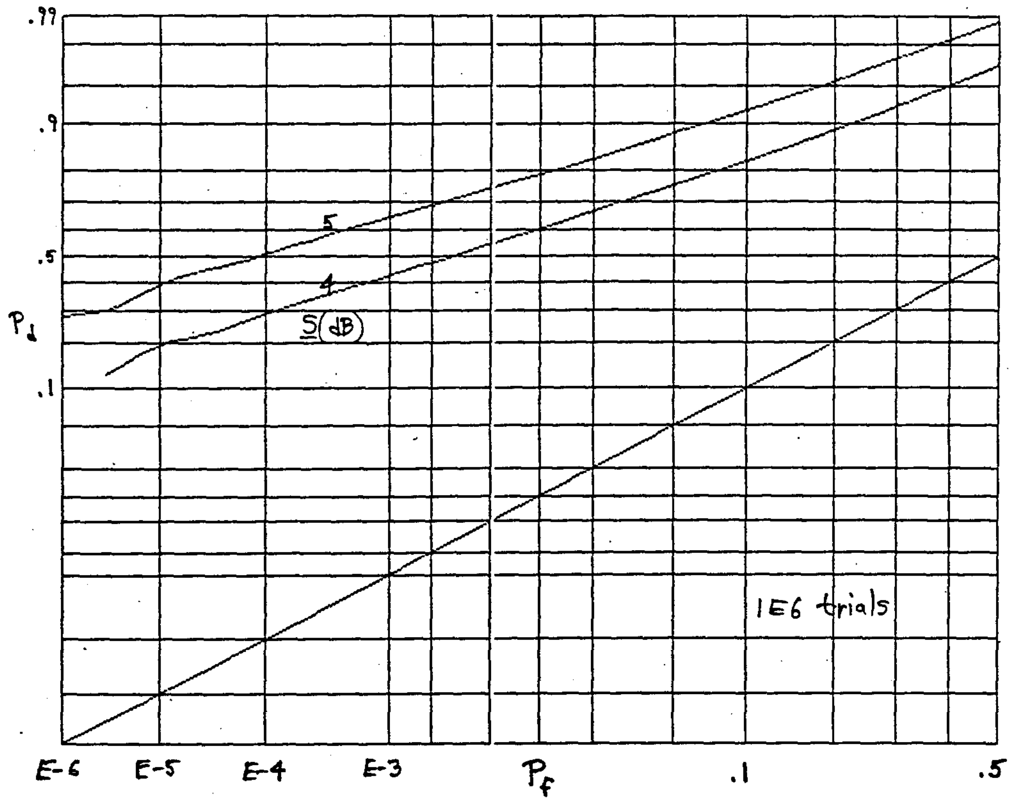

Figure 6. ROC for OPT-B, M = 16, N = 1024, $P_d \sim 0.5$

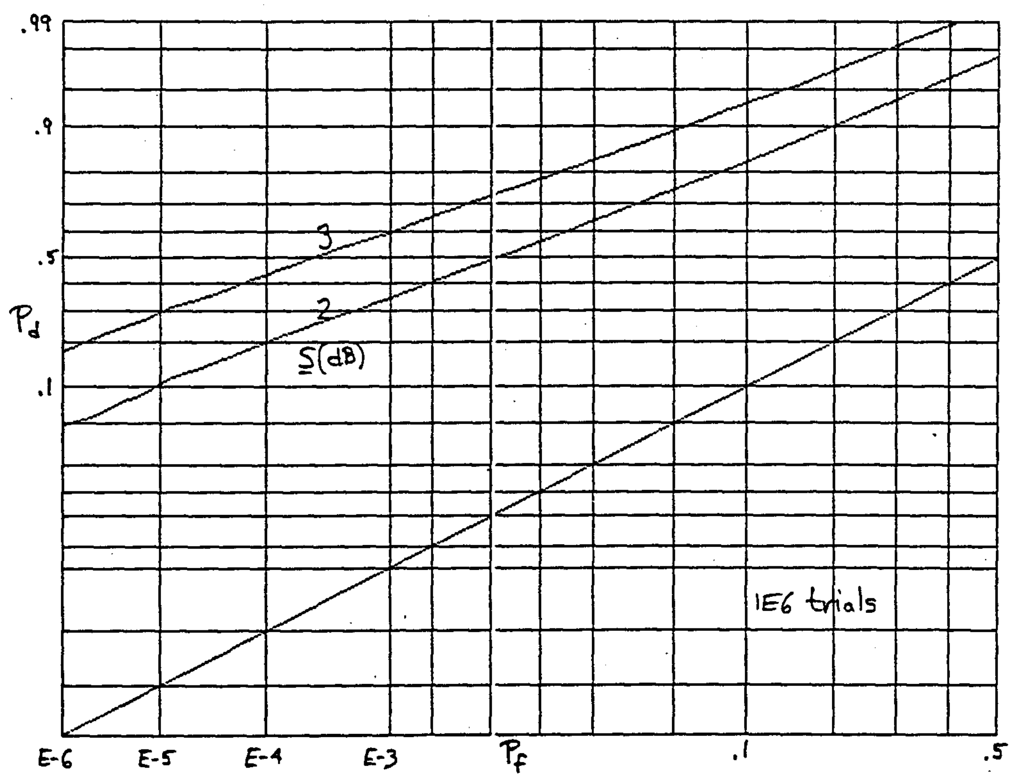

Figure 7. ROC for OPT-B, M = 32, N = 1024, $P_d \sim 0.5$

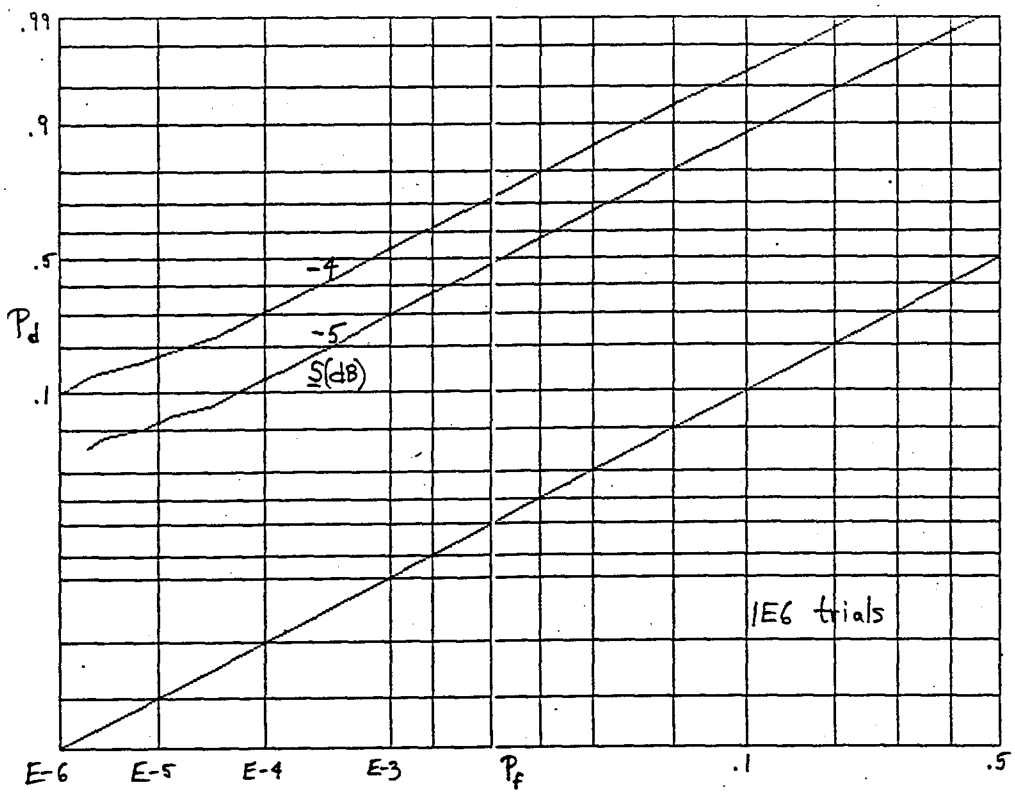

Figure 10. ROC for OPT-B, M = 256, N = 1024, $P_d \sim 0.5$

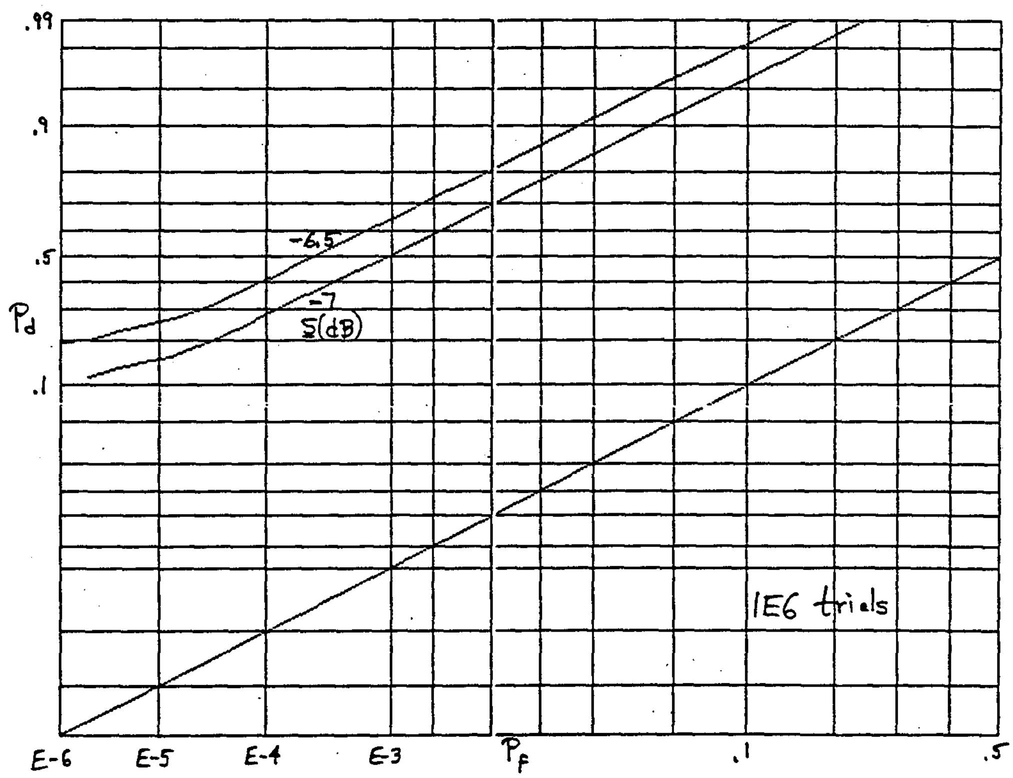

Figure 11. ROC for OPT-B, M = 512, N = 1024, $P_d \sim 0.5$

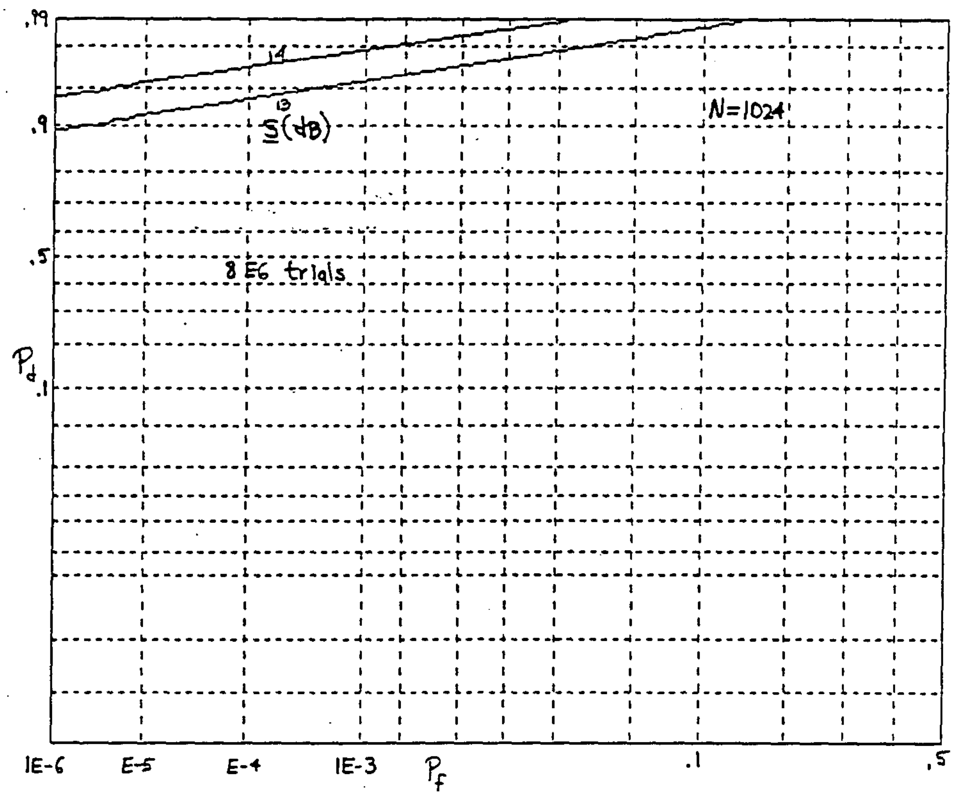

Figure 14. ROC for OPT-B, $\underline{M}$ = 4, N = 1024, P_d ~ 0.9

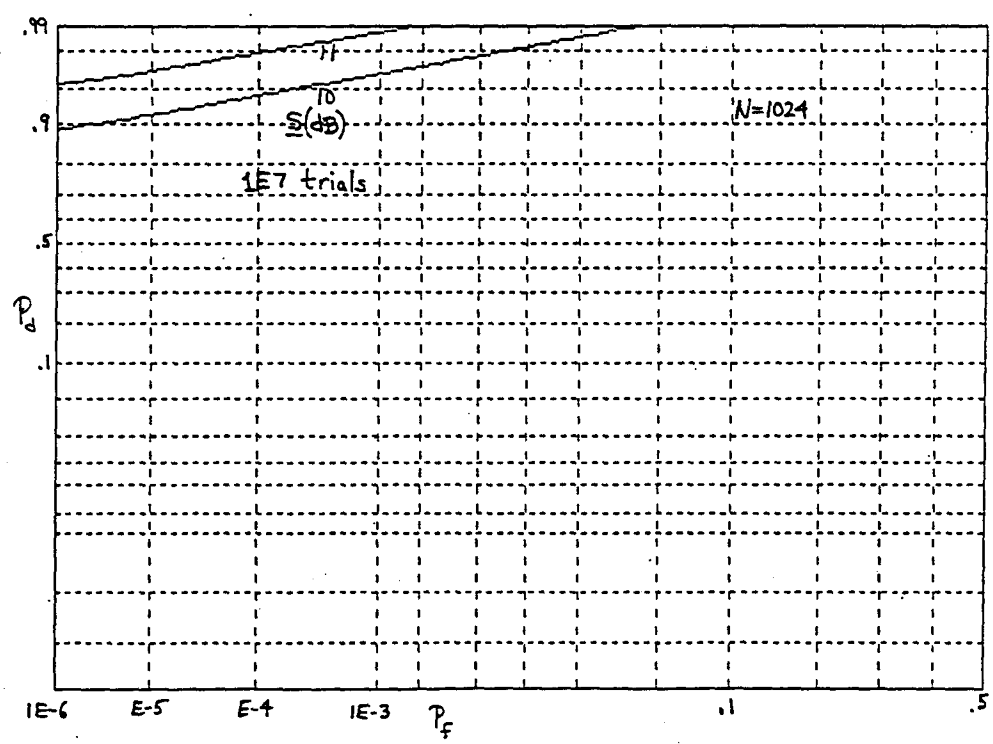

Figure 15. ROC for OPT-B, $\underline{M}$ = 8, N = 1024, P_d ~ 0.9

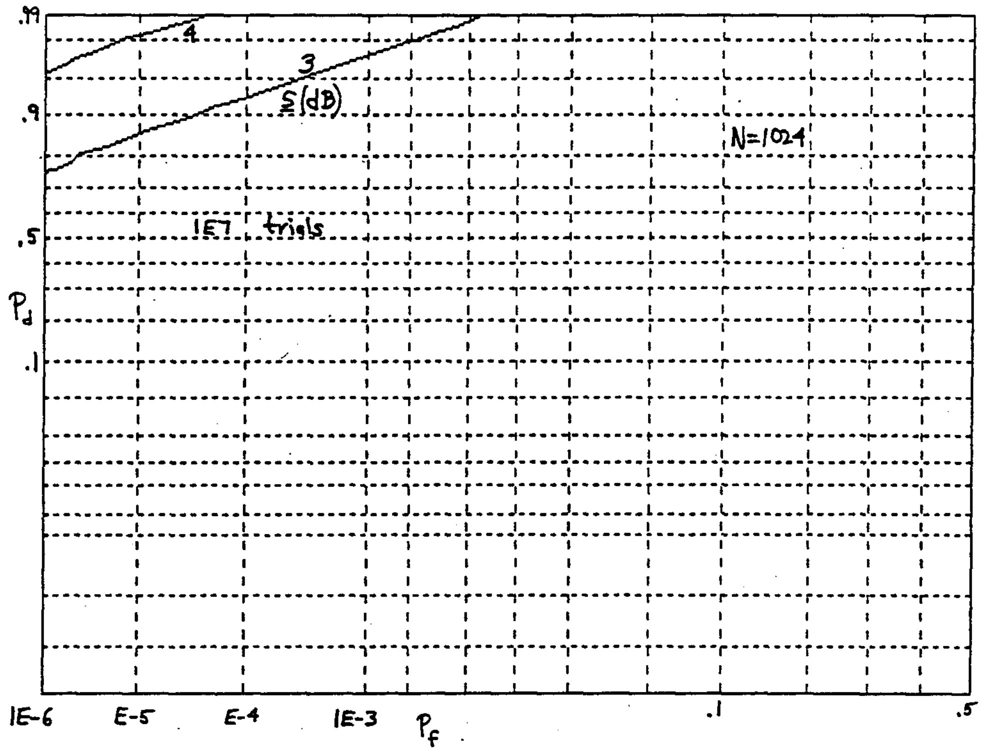

Figure 18. ROC for OPT-B, $\underline{M}$ = 64, N = 1024, P_d ~ 0.9

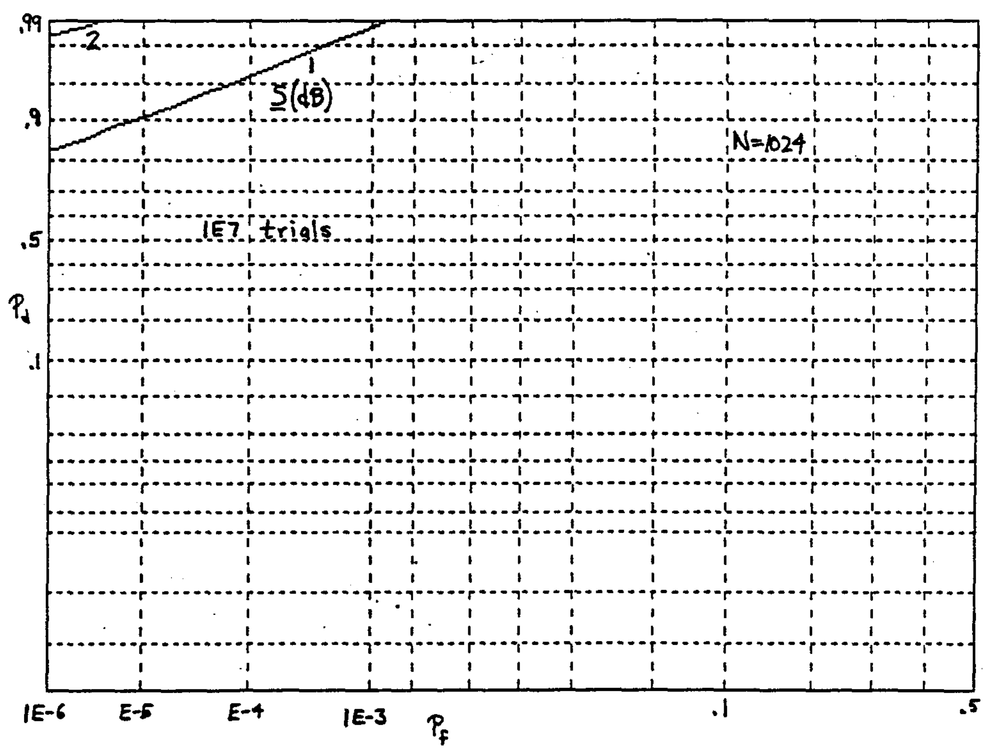

Figure 19. ROC for OPT-B, $\underline{M}$ = 128, N = 1024, P_d ~ 0.9

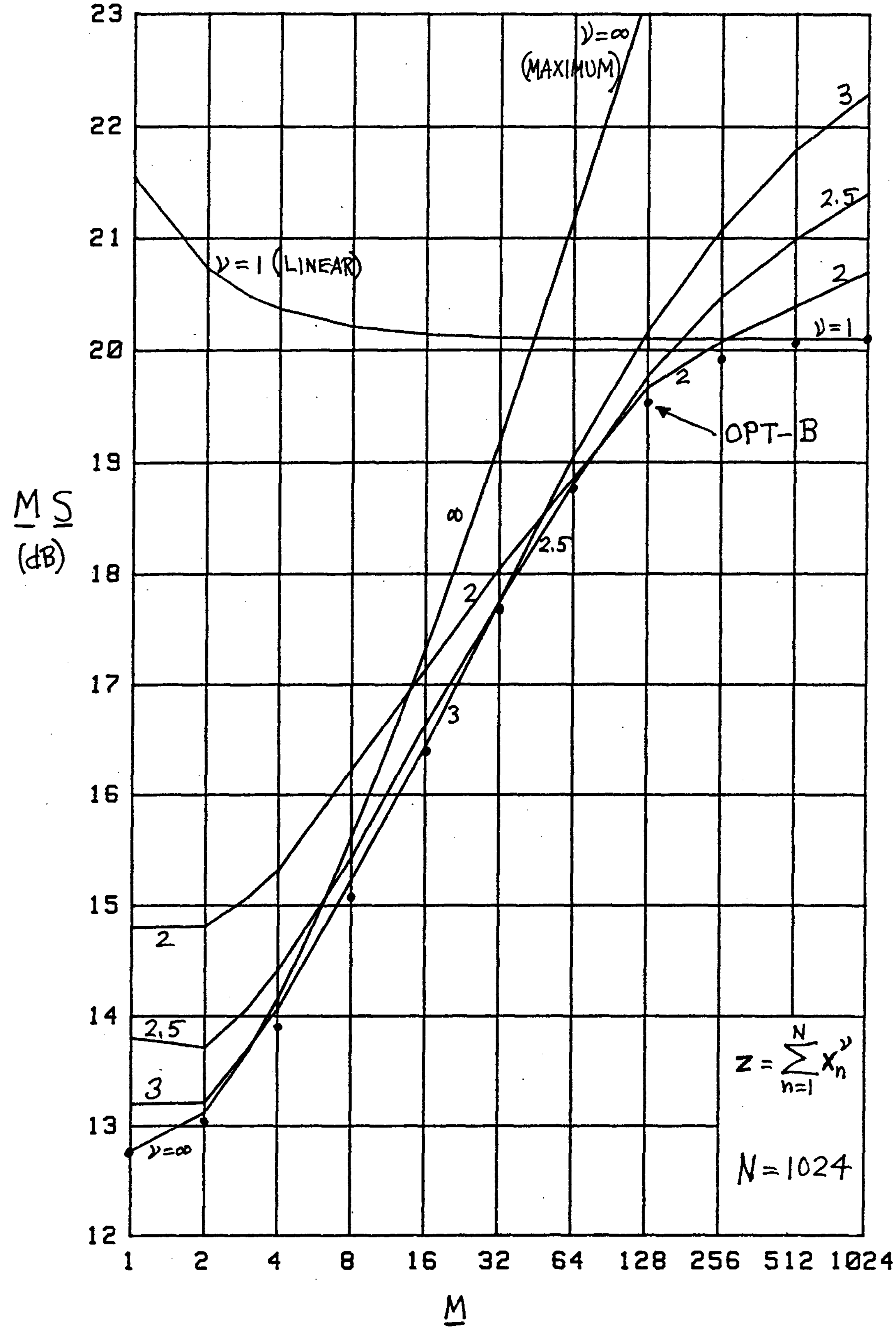

Figure 22. Total Signal Power Required for $P_f = 10^{-3}$, $P_d = 0.5$

COMPARISON WITH TWO OTHER PROCESSORS

MODIFIED GENERALIZED LIKELIHOOD RATIO PROCESSOR

The modified generalized likelihood ratio processor for search size N = 1024 was investigated in [1], resulting in 36 receiver operating characteristics for $\underline{M}$ = 8,16,32,64,128,256, and breakpoints x_o = 1,3,5,7,9,11; see [1; figures 8 - 43]. The remaining 36 receiver operating characteristics, for $\underline{M}$ = 1,2,3,4,512,1024 and the same 6 breakpoint values, are presented here in appendix B. A complete set of characteristics for x_o = 6 and $\underline{M}$ = 1,2,3,4,8,16,32,64,128,256,512,1024 has also been added. All these results were obtained by simulation; the number of trials utilized is indicated on each figure.

From this totality of results, the signal power $\underline{S}$ required to achieve specified false alarm and detection probabilities can be extracted. These results are presented in figure 24 for the low-quality operating point. They indicate that the best single compromise breakpoint x_o for the modified generalized likelihood ratio processor, in order to cover all $\underline{M}$ values from 1 to 1024, is x_o = 5.6; the maximum losses at $\underline{M}$ = 1 and 1024 are 3.7 dB relative to the OPT-B. This is considerably poorer than the power-law processor with power ν = 2.4, for which the maximum loss is only 1.2 dB, regardless of the value of $\underline{M}$; see figure 22.

LINEAR DEVICE WITH DEAD ZONE

The power-law processor was derived as an approximation to the optimum processor, while the modified generalized likelihood ratio processor was the result of an ad hoc change incorporated to circumvent weak signal estimates. These approaches are adopted, of necessity, in order to obtain realistic practical processors. However, they leave open the possibility of yet other suboptimum approaches that may yield equally good practical techniques for detection of weak signals of unknown character.

In this subsection, we consider a processor that comes about as a modification to the likelihood ratio test. The derivation of this processor (as well as its performance in terms of P_f and P_d) is given in appendix C. The end result is the following modified likelihood ratio test, (C-7), for signal presence:

$$\sum_{n=1}^{N} h(x_n) \gtrless v \ , \qquad h(x) = \begin{Bmatrix} x - b & \text{for} & x \geq b \\ 0 & \text{for} & x < b \end{Bmatrix} , \quad b \geq 0 \ . \tag{53}$$

The component device h(x) is linear for x > b, but has zero response for x < b. This device completely suppresses weak data values x_n, but allows the stronger data values to contribute linearly. Selection of the best choice of breakpoint b requires detailed computation of P_f and P_d for numerous values of $\underline{M}$ and $\underline{S}$.

A closed form for false alarm probability P_f is given in (C-13). It was used to generate the series of curves for P_f versus threshold v in figure 25. On the other hand, for the signal present hypothesis H_1, a closed form for the characteristic function of the decision variable is given by (C-14). This latter result was not used here for determining P_d; rather, P_d was found by simulation, since its most useful range of values is easily estimated.

A series of receiver operating characteristics were determined for processor (53), but are not presented here, for the sake of brevity. Tables of the signal powers per bin, $\underline{S}$, required for this modified likelihood ratio (MLR) processor (53) were extracted from these figures for both the low-quality and high-quality operating points; they are presented under the third column labeled MLR in tables 3 and 4 below. The last column gives the corresponding best value of breakpoint b to use. Also listed are the corresponding signal powers required for the OPT-B. The largest discrepancies in tables 3 and 4 between the OPT-B and MLR processors are approximately 0.35 dB, when the best breakpoint b is used.

This modified likelihood ratio processor needs to know signal size $\underline{M}$, in order to make the best selection of breakpoint b; this unrealistic condition is consistent with all the processors encountered. However, we are unable to make a decision on the best single compromise breakpoint b to use for this MLR processor, because the large number of receiver operating characteristics that would be required to cover all the potential b and $\underline{M}$ values were not run. The analytic results in appendix C would be useful in this regard.

SUMMARY

The limiting detection performance for random signals of unknown location, structure, extent, and strength has been very accurately ascertained through use of a "super" optimum processor called the optimum banded processor. This processor presumes knowledge of the number $\underline{M}$ of occupied bins, the signal power per bin $\underline{S}$, and some partial knowledge about the locations of the occupied bins. This unrealistic set of assumptions leads to an upper bound on attainable detectability performance of any practical processor; however, the bound turns out to be so tight that searching for further improvement can yield only negligible gains, of the order of 0.1 dB.

The tightness of the upper bound is established by comparing it with the performance of the power-law processor using its best power ν. It is found that there is always some power-law processor (some value of ν) that performs within 0.1 dB of the absolute optimum level. Strictly, this conclusion is drawn only for the numerical example of $N = 1024$, $P_f = 10^{-3}$, $P_d = 0.5$. This achievement for the power-law processor is striking, in light of the fact that it knows and uses <u>nothing</u> about size $\underline{M}$, signal power $\underline{S}$, nor anything about the signal locations. However, when it is necessary to employ a single compromise value of power ν to operate in the <u>entire</u> range of possibilities from $\underline{M} = 1$ to $\underline{M} = N = 1024$, the minimax loss is 1.2 dB, occurring for power selection $\nu = 2.4$.

APPENDIX A. FALSE ALARM PROBABILITY FOR OPTIMUM BANDED PROCESSOR

We are interested in obtaining the false alarm probability for the processor described in (49) and (50), where

$$e_n = \exp(\underline{w}\ x_n)\ ,\quad \underline{w} = \frac{\underline{S}}{1+\underline{S}}\ ,\quad \omega = 1 + \frac{1}{\underline{w}} = 2 + \frac{1}{\underline{S}}\ ,\quad I = \frac{N}{\underline{M}}\ . \tag{A-1}$$

We define the auxiliary random variables

$$s_m = \sum_{j=1}^{I} e_{mI-I+j}\ ,\quad y_m = \ln(s_m) \quad \text{for } 1 \le m \le \underline{M}\ , \tag{A-2}$$

$$z = y_1 + \cdots + y_{\underline{M}}\ . \tag{A-3}$$

The random variable s_m comes directly from (50), while z is the logarithm of the left-hand side of (49). Since random variable x_n is never negative here, we always have $e_n \ge 1$, $s_m \ge I$, $y_m \ge \ln(I)$, $z \ge \underline{M}\ln(I)$. Also, $\omega \ge 2$.

The fundamental calculation is the characteristic function of e_n, which for $\xi > 0$, can be developed under H_0 as

$$f_e(\xi) \equiv \overline{\exp\left(i\xi e_n\right)} = \overline{\exp\left(i\xi\ \exp(\underline{w}x_n)\right)} =$$

$$= \int_0^\infty du\ \exp(-\ u + i\xi\ \exp(\underline{w}u)) = (\omega-1)\int_1^\infty dt\ t^{-\omega}\exp(i\xi t) = \tag{A-4}$$

$$= (\omega-1)(-i\xi)^{\omega-1}\int_{-i\xi}^{-i\infty} dy\ y^{-\omega}\exp(-y) = (\omega-1)(-i\xi)^{\omega-1}\int_{-i\xi}^{\infty} dy\ y^{-\omega}\exp(-y)$$

$$= (\omega-1)\ (-i\xi)^{\omega-1}\ [\Gamma(1-\omega) - \gamma(1-\omega,-i\xi)] =$$

If $f_e(\xi)$ is evaluated via (A-5) and (A-6) at increment $\Delta\xi_1$, then $f_s(\xi)$ becomes available at the same increment. Use of these samples in (A-7) yields an approximation $\tilde{p}_s(u)$ with period $2\pi/\Delta\xi_1$, which must be large enough to encompass the full extent of $p_s(u)$. If an FFT (fast Fourier transform) of size N_1 is used to carry out (A-7), the increment for the argument of $\tilde{p}_s(u)$ is $\Delta u_1 = 2\pi/(N_1\ \Delta\xi_1)$; since Δu_1 must be small, this will require a large value for N_1.

If $f_y(\xi)$ in (A-8) is evaluated at increment $\Delta\xi_2$, then $f_z(\xi)$ is available at the same increment. When these samples are used in the FFT required to determine the density $p_z(u)$ of z, the approximation $\tilde{p}_z(u)$ has period $2\pi/\Delta\xi_2$, which must be large. If this FFT is taken at size N_2, the increment for the argument of $\tilde{p}_z(u)$ is $\Delta u_2 = 2\pi/(N_2\ \Delta\xi_2)$, which must be small enough to track the variations.

The fundamental parameters in this investigation are N, $\underline{M}$, and $\underline{S}$, where $I = N/\underline{M}$ must be an integer. Two different numerical integrations are needed to carry out (A-6) and (A-8). Also, two FFTs are required: one for $\tilde{p}_s(u)$ from $f_s(\xi)$ in (A-7) and the other for $\tilde{p}_z(u)$ from $f_z(\xi)$. This intensive numerical procedure will require large values for FFT sizes N_1 and N_2.

APPENDIX B. RECEIVER OPERATING CHARACTERISTICS FOR MODIFIED GENERALIZED LIKELIHOOD RATIO PROCESSOR

The modified generalized likelihood ratio (MGLR) processor for search size N = 1024 was investigated in [1], resulting in 36 receiver operating characteristics for $\underline{M}$ = 8,16,32,64,128,256, and breakpoints x_o = 1,3,5,7,9,11. The remaining 36 receiver operating characteristics at N = 1024, for $\underline{M}$ = 1,2,3,4,512,1024 and the same 6 breakpoint values, are presented here. Also, a complete set of characteristics for x_o = 6 and $\underline{M}$ = 1,2,3,4,8,16, 32,64,128,256,512,1024 has been added. Finally, a couple of isolated examples for x_o = 13 and x_o = 15 have been run.

All these results were obtained by simulation; the number of trials utilized is indicated on each figure. A couple of check cases were conducted for comparison with the analytical results in [1]. They are x_o = 3, $\underline{M}$ = 256 in figure B-11, which verifies [1; figure 39], and x_o = 7, $\underline{M}$ = 8 in figure B-36, which verifies [1; figure 11]. Labeling on all plots is identical to figure B-1.

A short table of losses relative to the OPT-B, for N = 1024, $P_f = 10^{-3}$, $P_d = 0.5$, is presented below. These were extracted from the accompanying receiver operating characteristics.

x_o	dB loss at $\underline{M}$ = 1	dB loss at $\underline{M}$ = 1024
1	5.5	0.5
3	5.4	1.3
5	4.2	3.1
6	3.4	4.1
7	2.7	5.0
9	1.2	>7
11	0	—
13	0	—

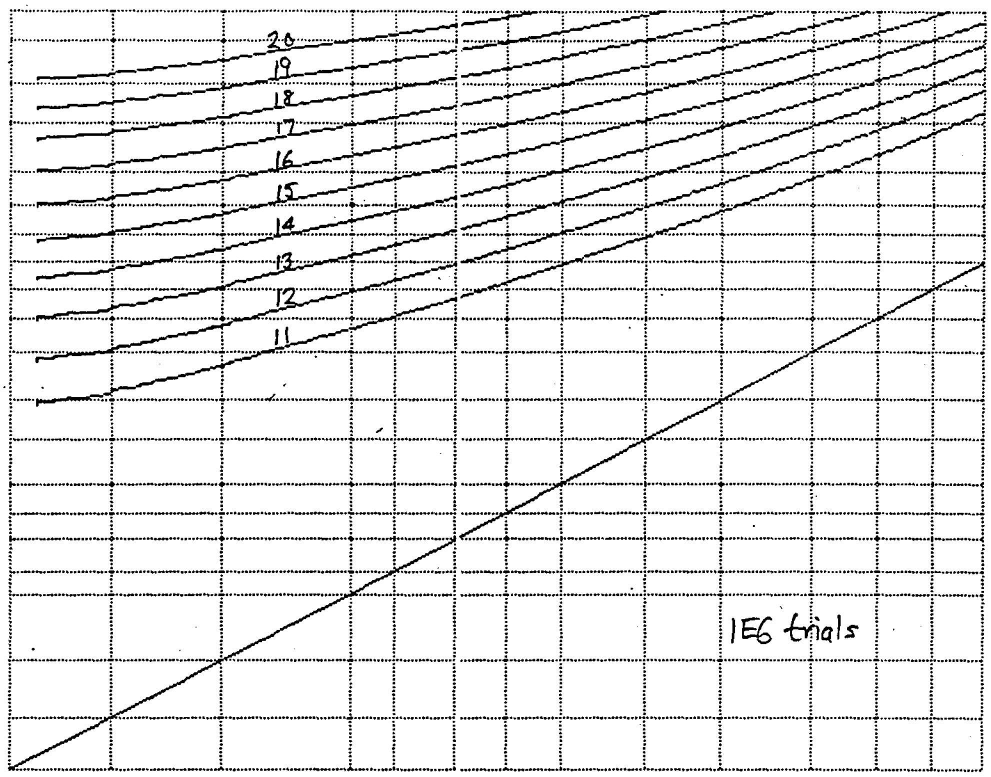

Figure B-3. MGLR ROC for $x_o = 1$, $\underline{M} = 3$, $N = 1024$

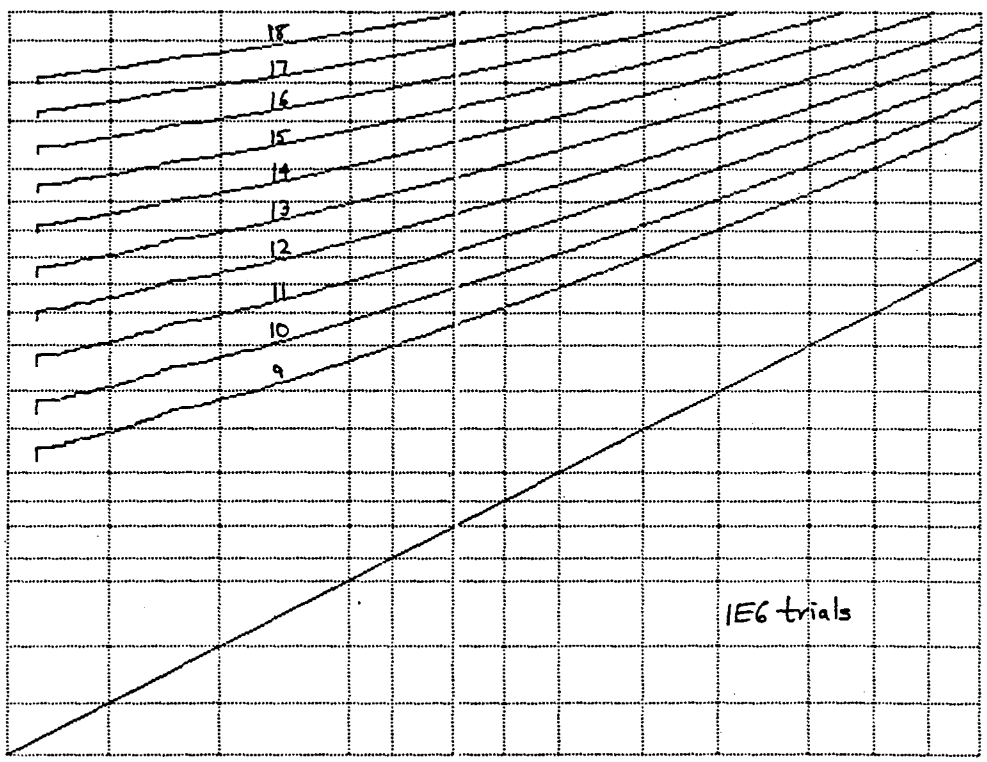

Figure B-4. MGLR ROC for $x_o = 1$, $\underline{M} = 4$, $N = 1024$

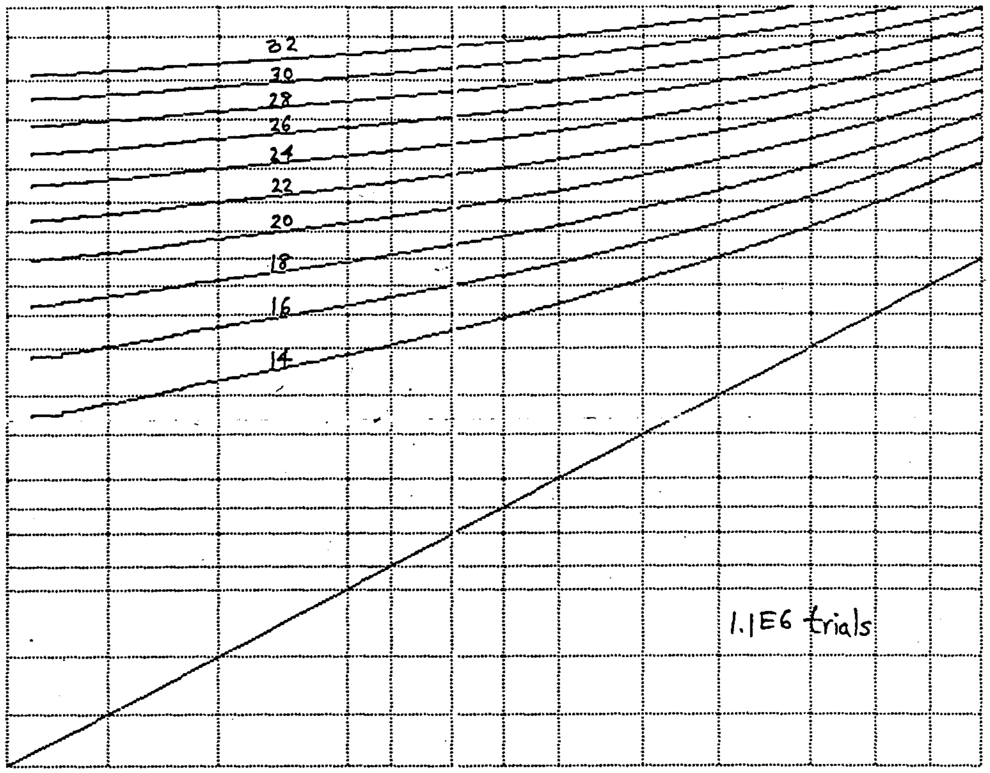

Figure B-7. MGLR ROC for $x_o = 3$, $\underline{M} = 1$, $N = 1024$

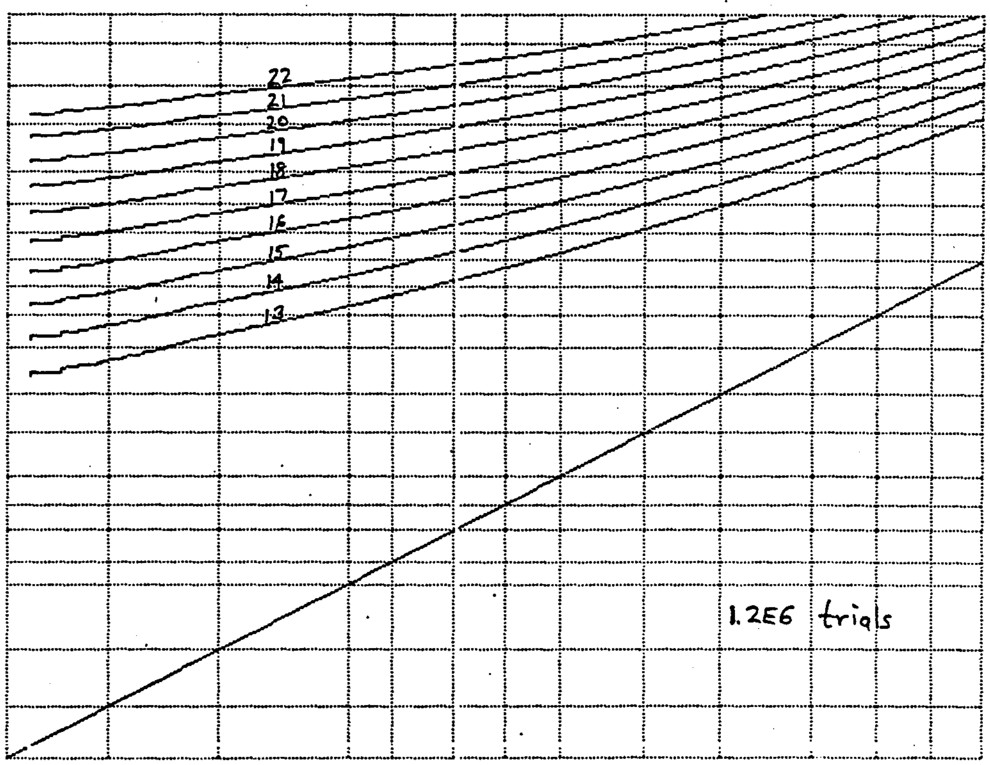

Figure B-8. MGLR ROC for $x_o = 3$, $\underline{M} = 2$, $N = 1024$

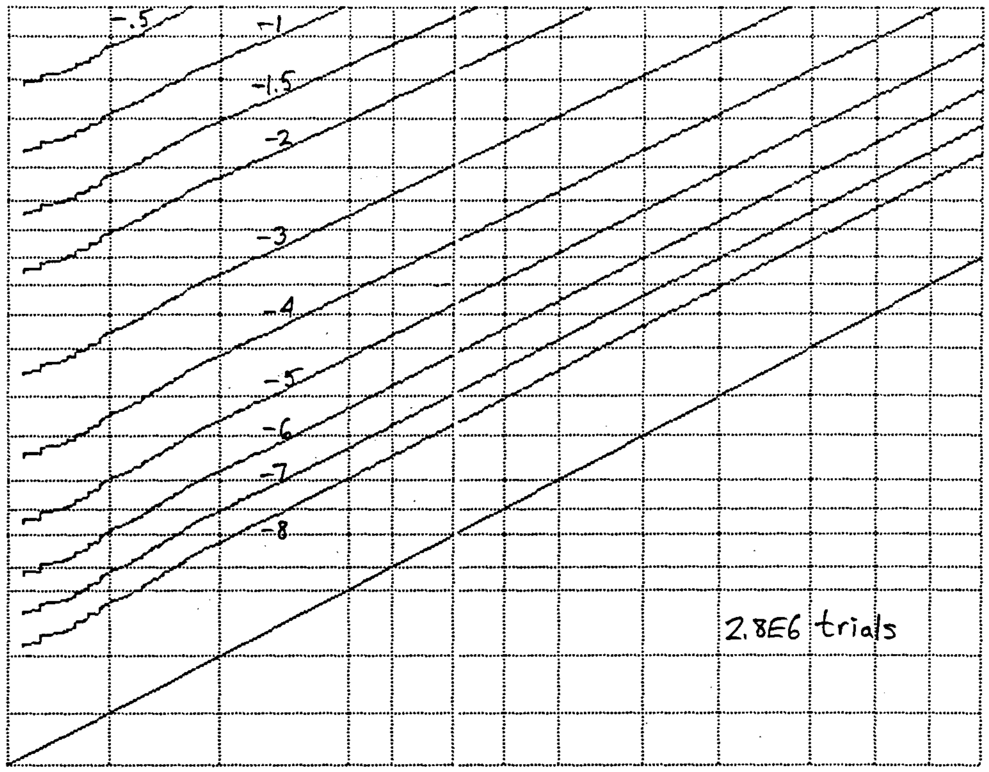

Figure B-11. MGLR ROC for $x_o = 3$, $\underline{M} = 256$, $N = 1024$

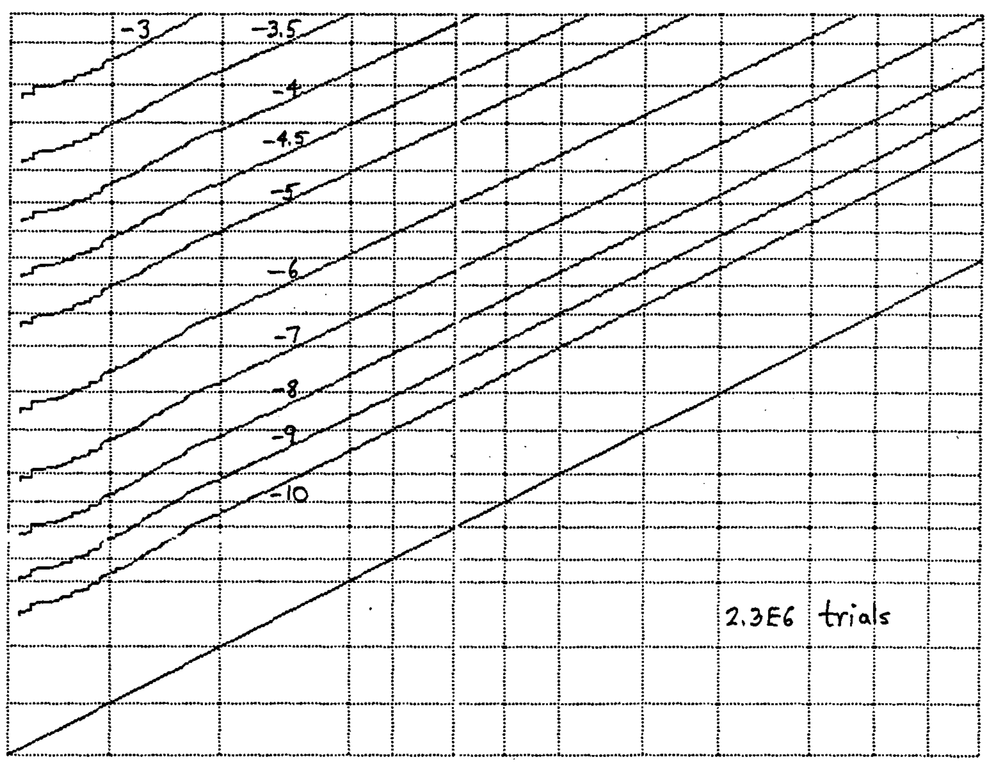

Figure B-12. MGLR ROC for $x_o = 3$, $\underline{M} = 512$, $N = 1024$

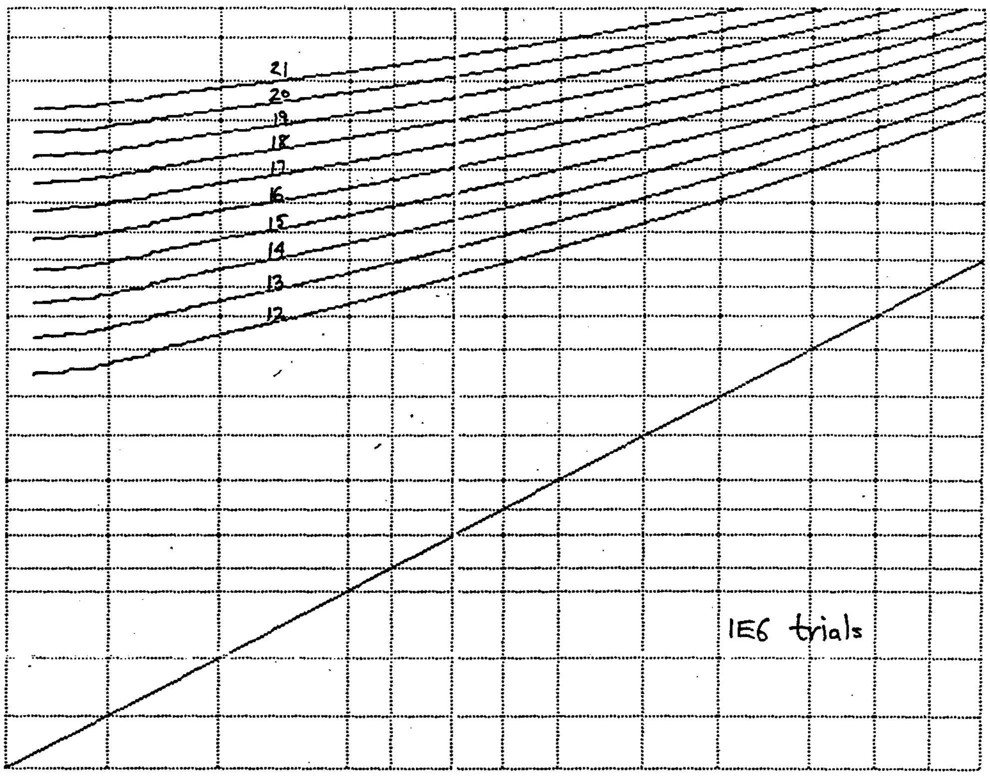

Figure B-15. MGLR ROC for $x_o = 5$, $\underline{M} = 2$, N = 1024

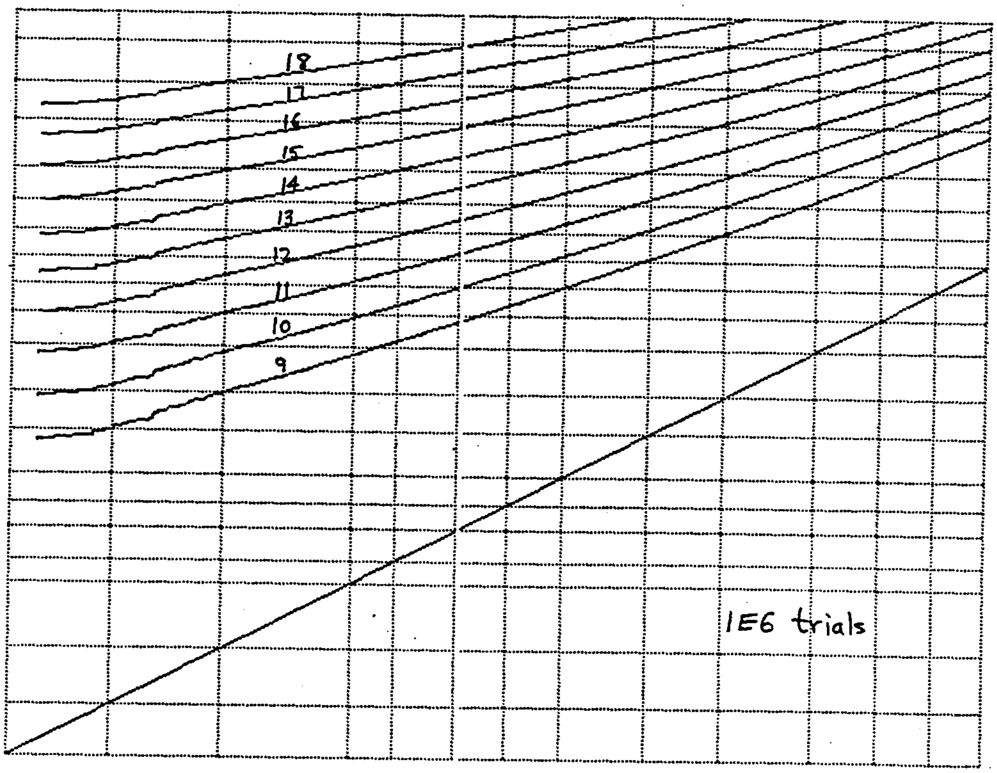

Figure B-16. MGLR ROC for $x_o = 5$, $\underline{M} = 3$, N = 1024

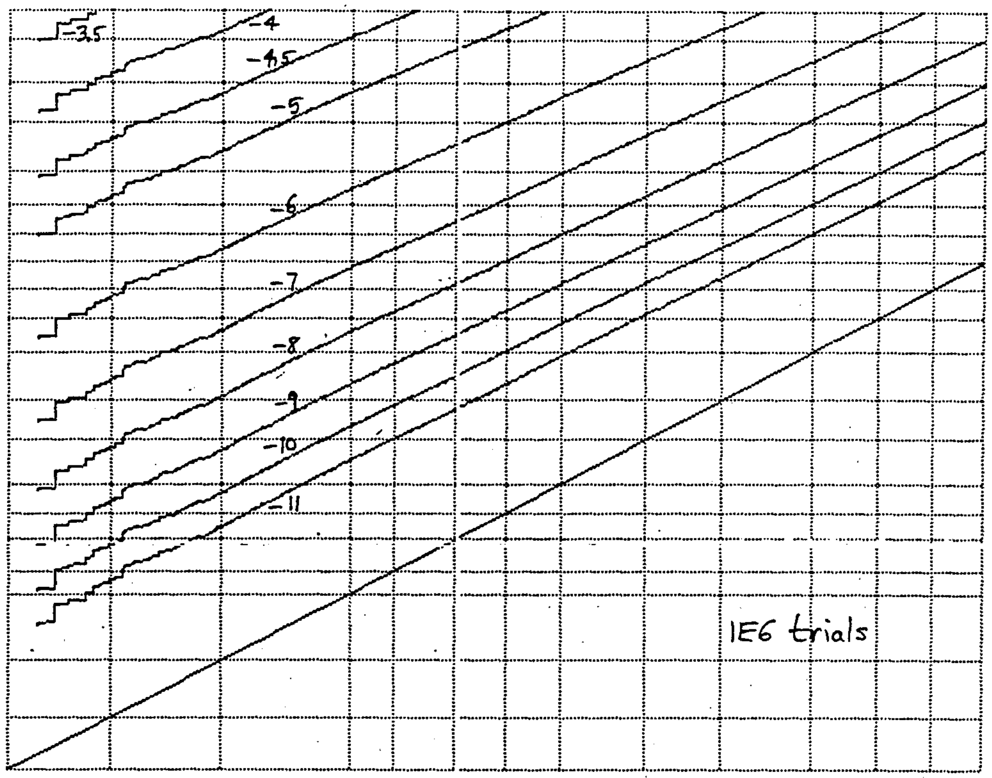

Figure B-19. MGLR ROC for $x_o = 5$, $\underline{M} = 1024$, $N = 1024$

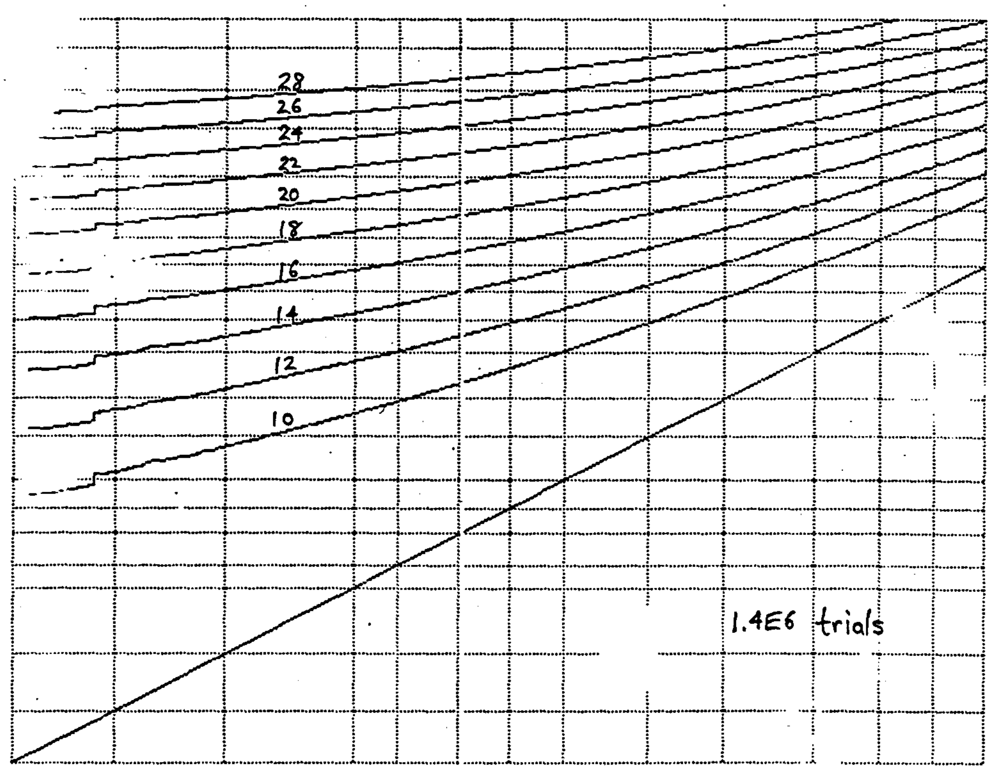

Figure B-20. MGLR ROC for $x_o = 6$, $\underline{M} = 1$, $N = 1024$

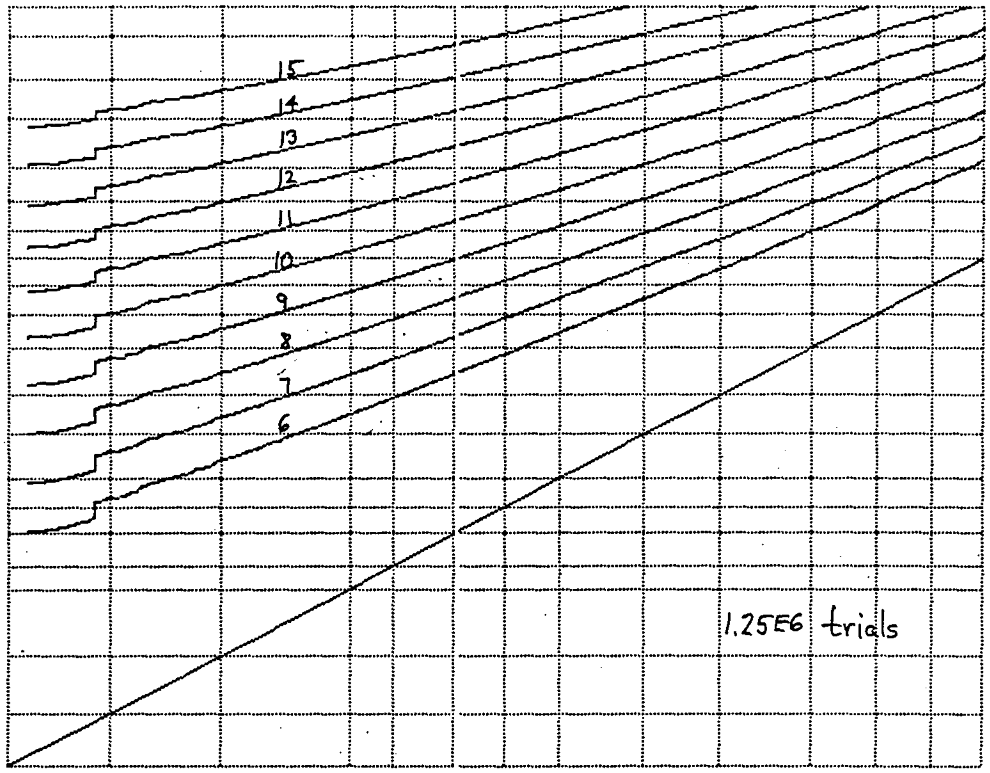

Figure B-23. MGLR ROC for $x_o = 6$, $\underline{M} = 4$, $N = 1024$

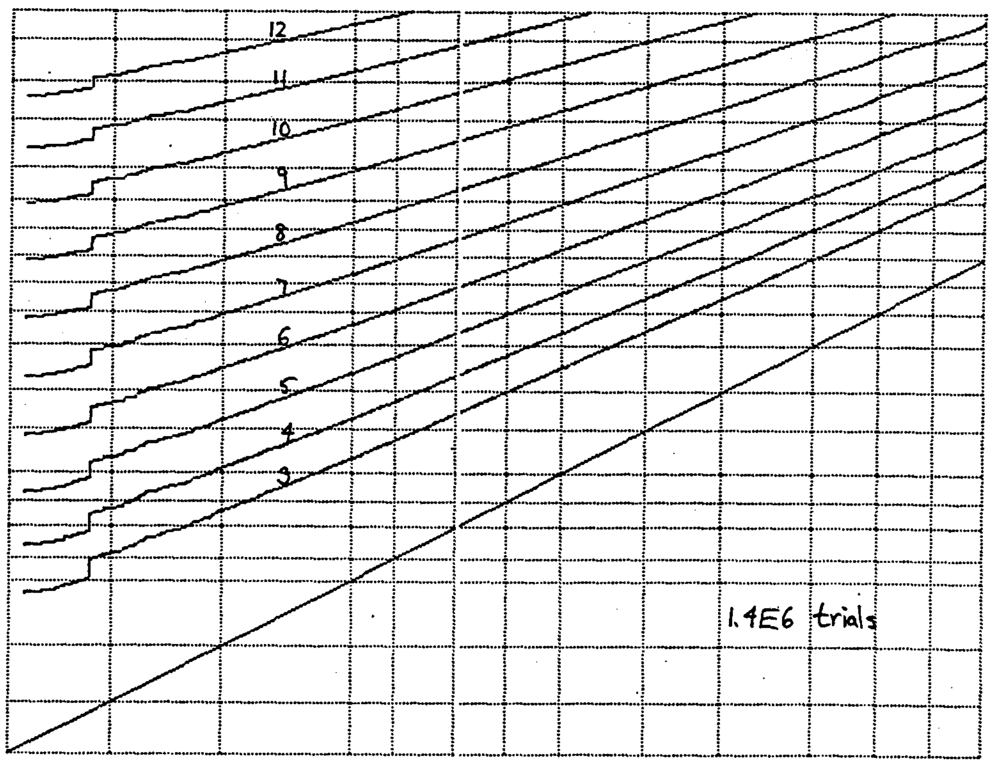

Figure B-24. MGLR ROC for $x_o = 6$, $\underline{M} = 8$, $N = 1024$

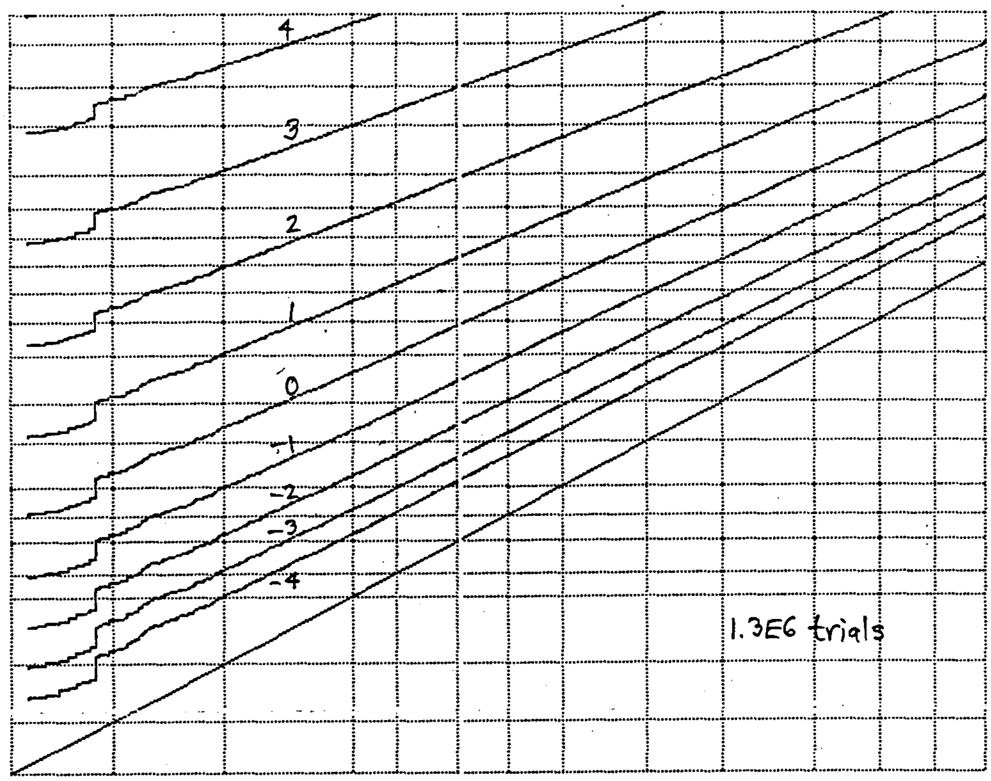

Figure B-27. MGLR ROC for $x_o = 6$, $\underline{M} = 64$, N = 1024

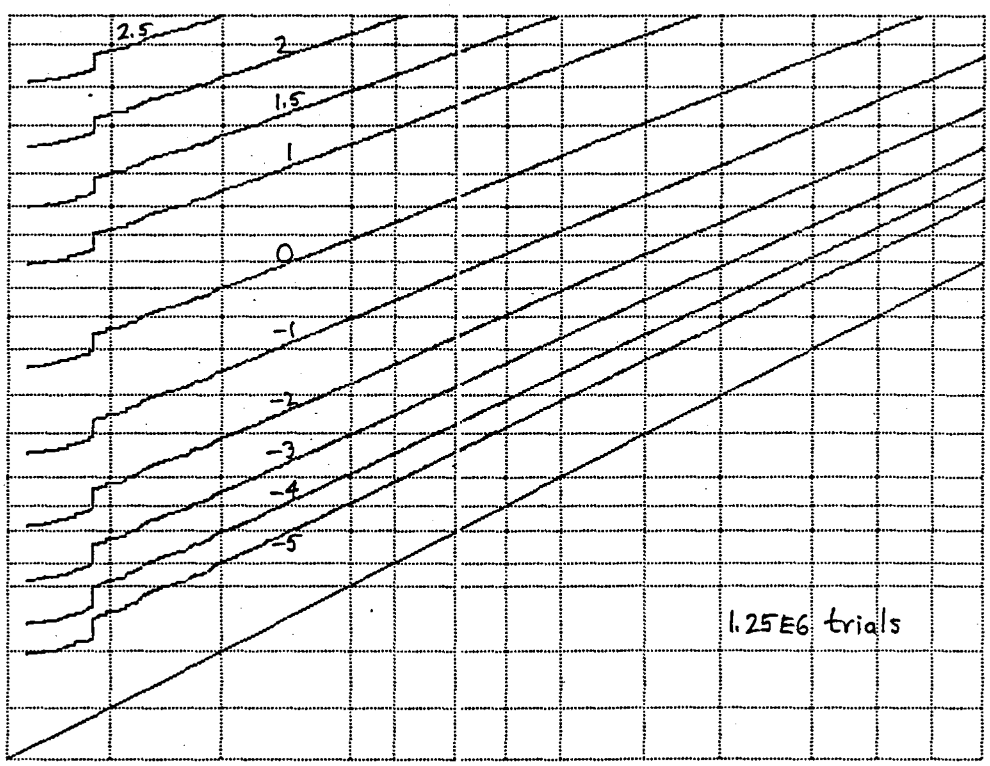

Figure B-28. MGLR ROC for $x_o = 6$, $\underline{M} = 128$, N = 1024

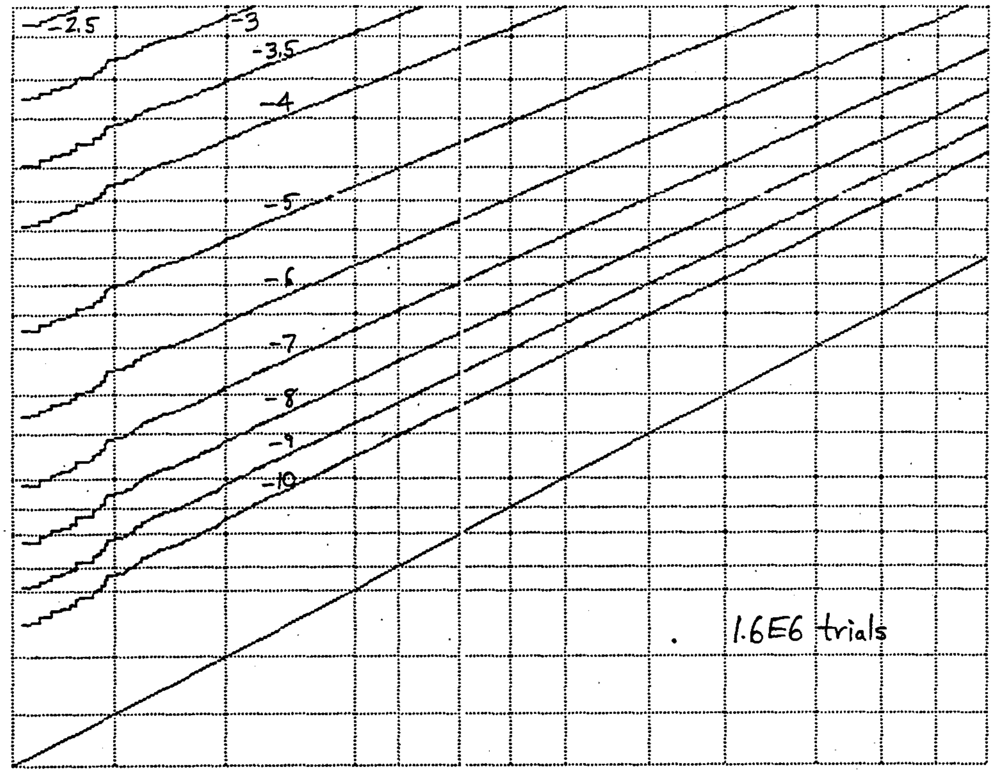

Figure B-31. MGLR ROC for x_o = 6, $\underline{M}$ = 1024, N = 1024

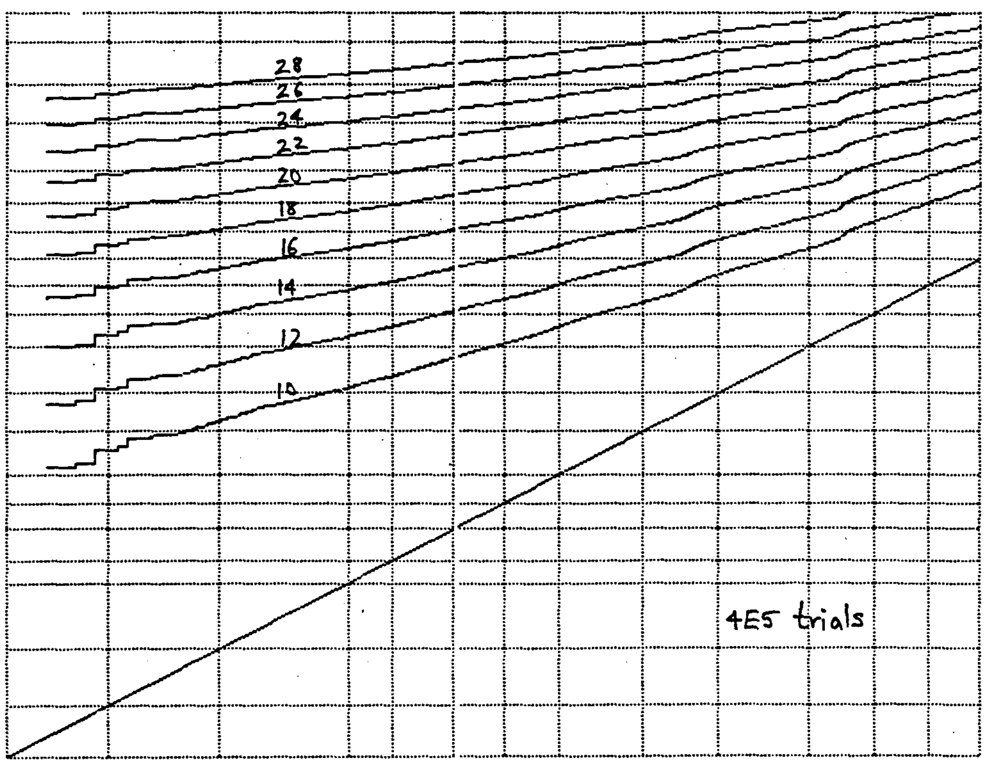

Figure B-32. MGLR ROC for x_o = 7, $\underline{M}$ = 1, N = 1024

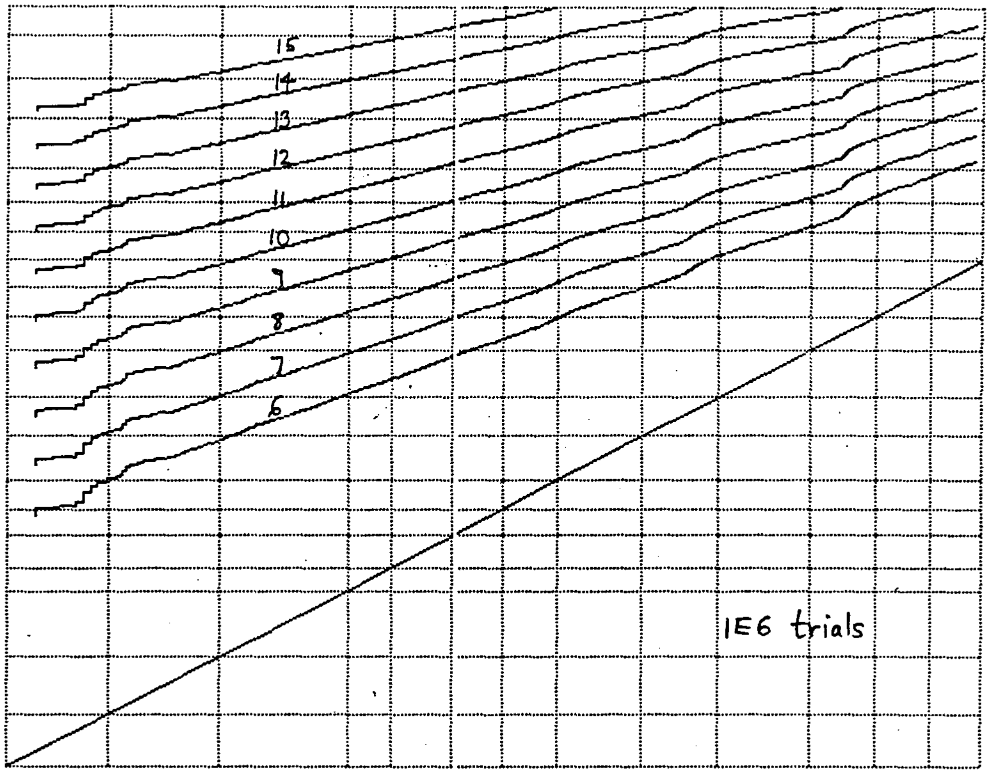

Figure B-35. MGLR ROC for $x_o = 7$, $\underline{M} = 4$, $N = 1024$

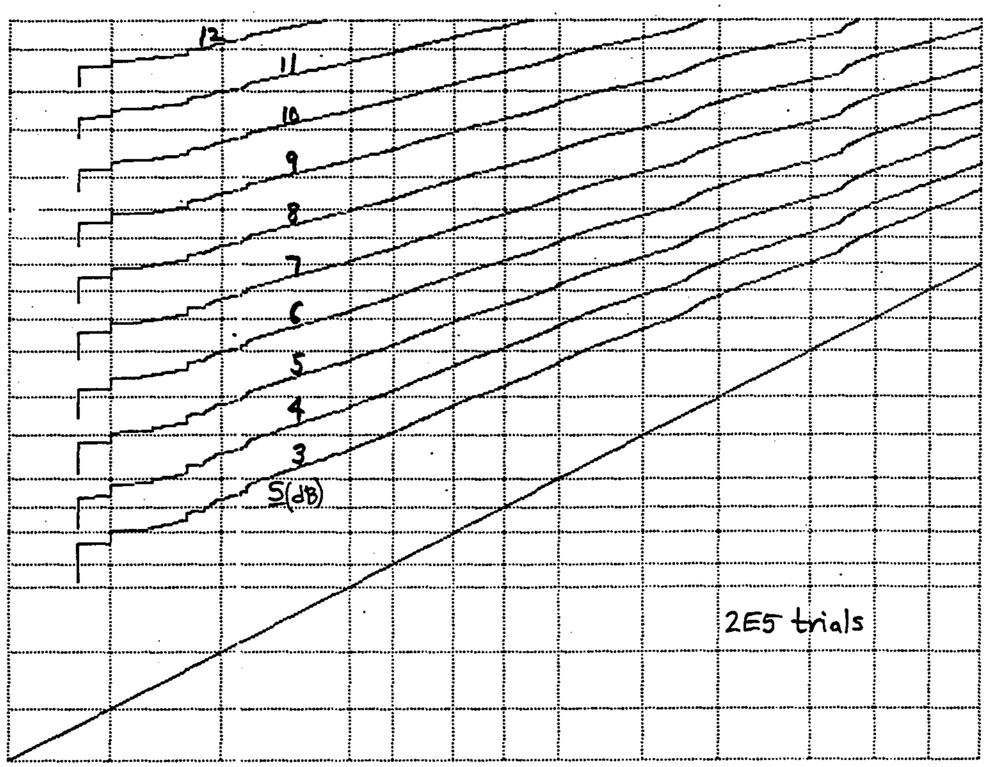

Figure B-36. MGLR ROC for $x_o = 7$, $\underline{M} = 8$, $N = 1024$

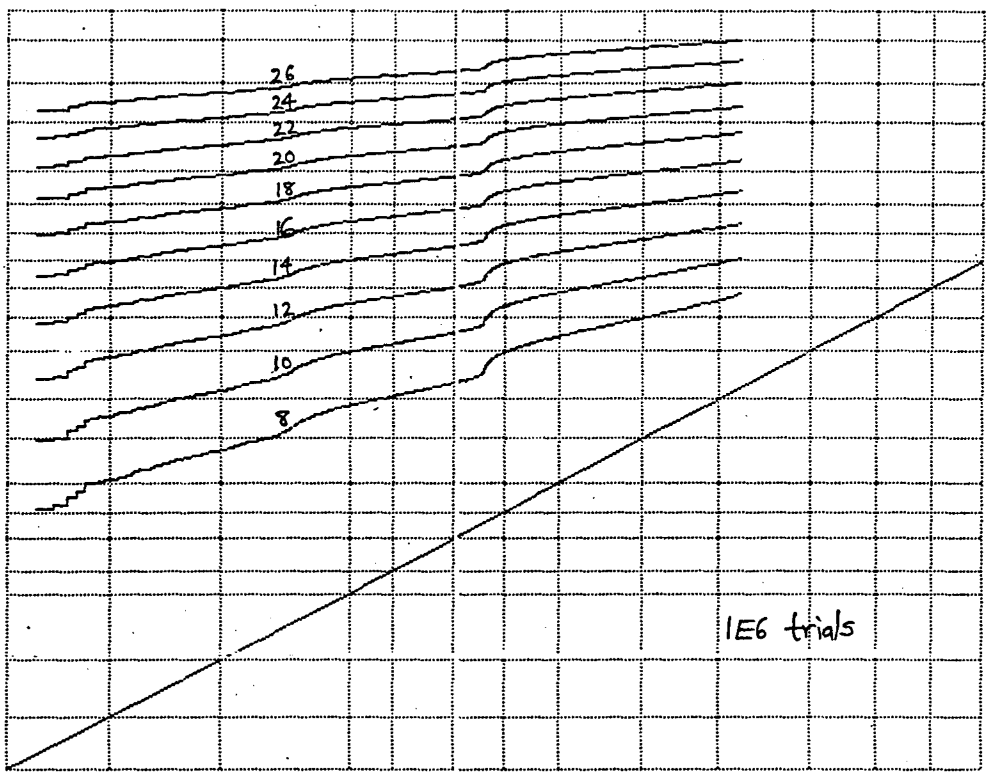

Figure B-39. MGLR ROC for $x_o = 9$, $\underline{M} = 1$, $N = 1024$

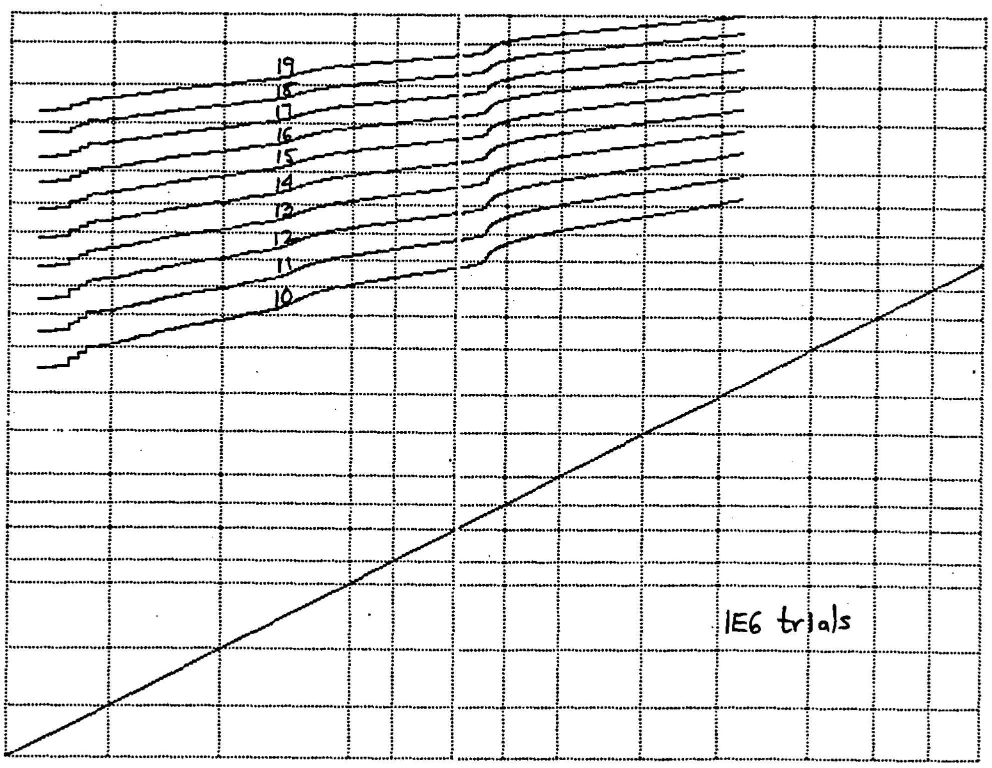

Figure B-40. MGLR ROC for $x_o = 9$, $\underline{M} = 2$, $N = 1024$

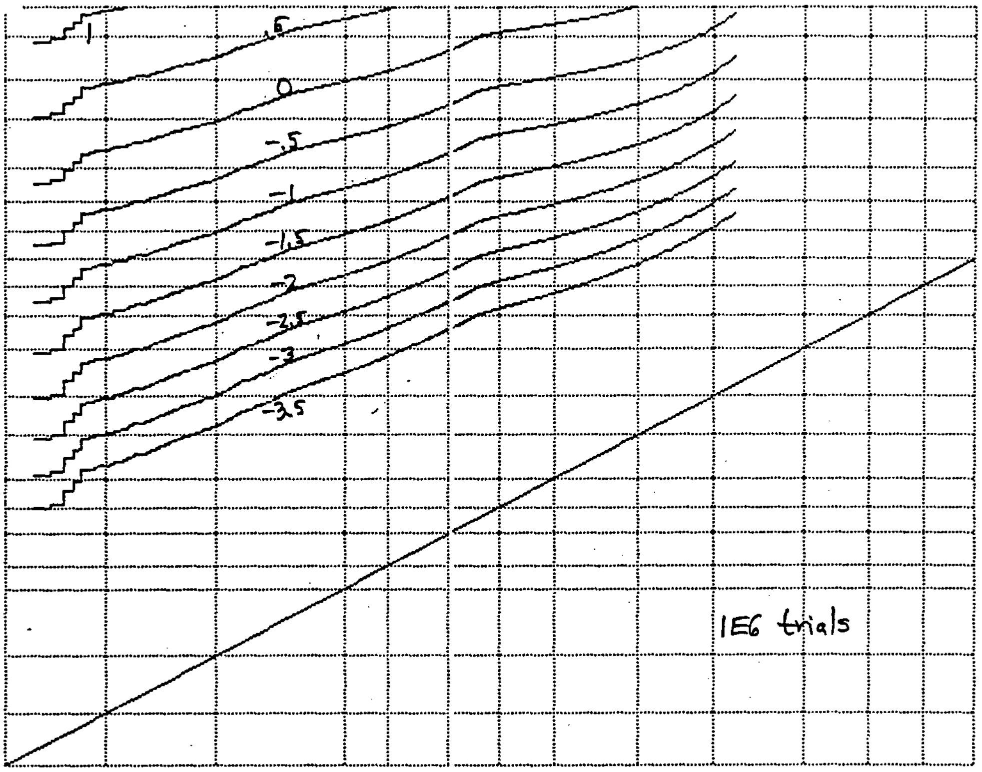

Figure B-43. MGLR ROC for $x_o = 9$, $\underline{M} = 512$, $N = 1024$

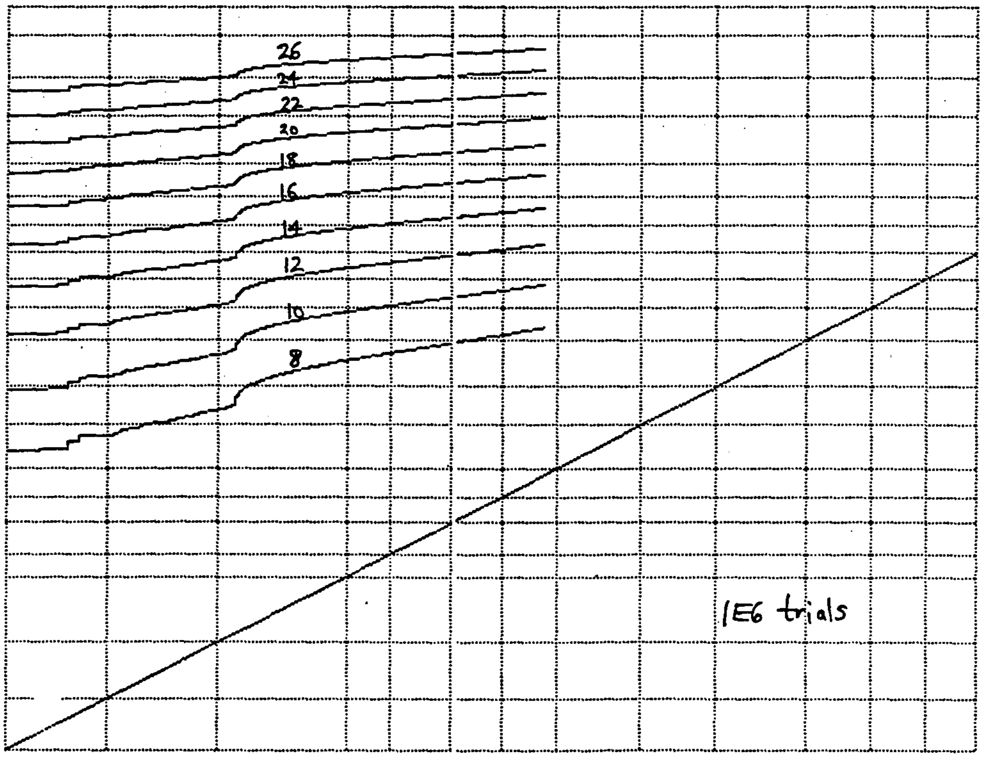

Figure B-44. MGLR ROC for $x_o = 11$, $\underline{M} = 1$, $N = 1024$

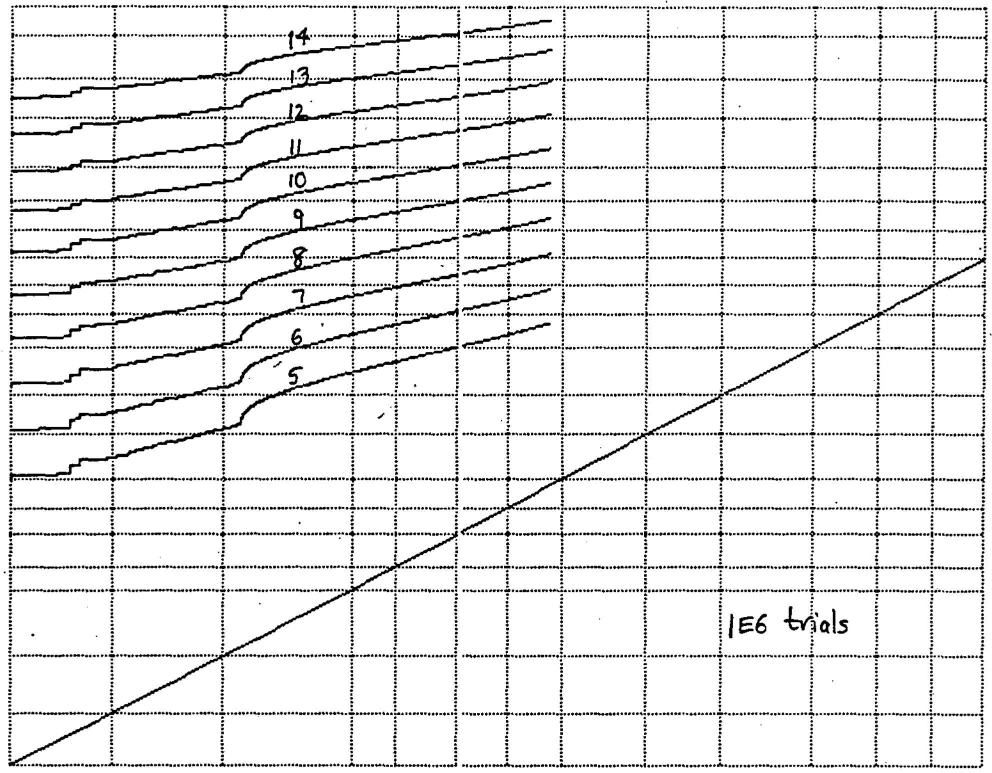

Figure B-47. MGLR ROC for $x_o = 11$, $\underline{M} = 4$, $N = 1024$

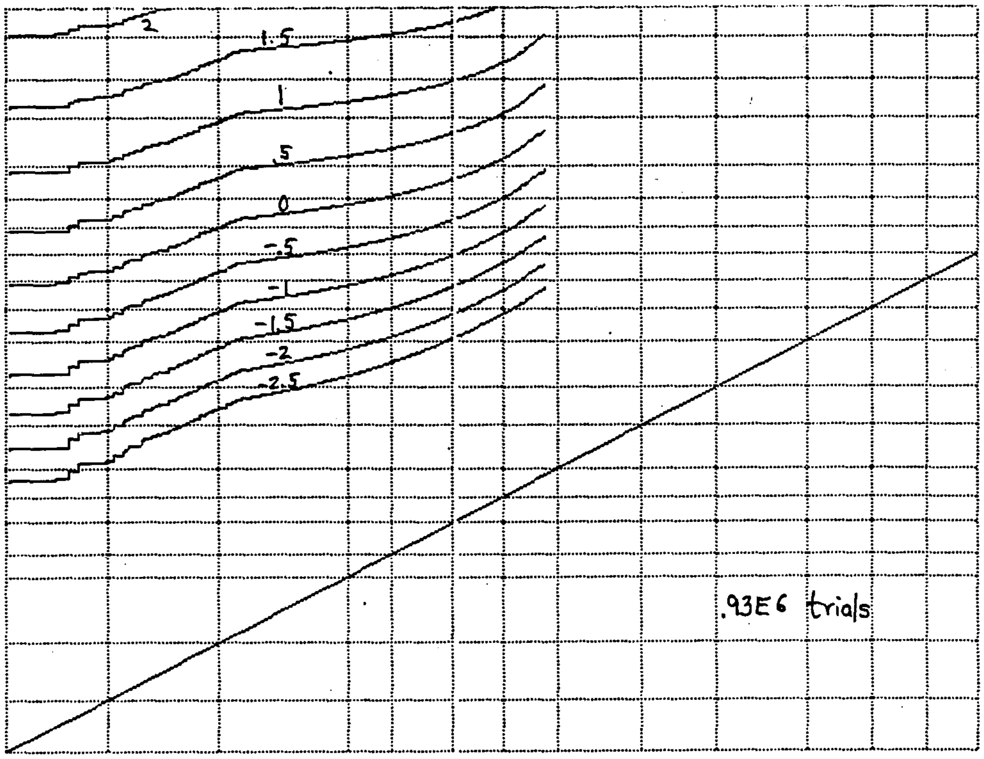

Figure B-48. MGLR ROC for $x_o = 11$, $\underline{M} = 512$, $N = 1024$

APPENDIX C. DERIVATION OF MODIFIED LIKELIHOOD RATIO TEST

For known (possibly unequal) signal powers $\{\underline{S}_n\}$, the likelihood ratio may be shown to take the form

$$\mathbf{LR} = \prod_{n=1}^{N} \left\{ \frac{1}{1+\underline{S}_n} \exp\left(\frac{\underline{S}_n}{1+\underline{S}_n}\, x_n \right) \right\}, \quad \underline{S}_n \ge 0 , \tag{C-1}$$

leading to the likelihood ratio test

$$\sum_{n=1}^{N} \frac{\underline{S}_n}{1+\underline{S}_n}\, x_n \gtrless v . \tag{C-2}$$

However, if the signal powers are unknown, we replace $\underline{S}_n$ by hypothesized value S_n, obtaining the modified likelihood ratio

$$\mathbf{MLR} \equiv \prod_{n=1}^{N} \left\{ \frac{1}{1+S_n} \exp\left(\frac{S_n}{1+S_n}\, x_n \right) \right\}, \quad S_n \ge 0 . \tag{C-3}$$

This quantity is maximized by the choices (random variables)

$$S_n = \begin{Bmatrix} x_n - 1 & \text{for } x_n \ge 1 \\ 0 & \text{for } x_n < 1 \end{Bmatrix} \quad \text{for } 1 \le n \le N . \tag{C-4}$$

When these values are substituted into (C-3), there follows the generalized likelihood ratio test

$$\sum_{n=1}^{N} g(x_n) \gtrless v , \tag{C-5}$$

where $g(x)$ is given by (19). This is a familiar result [1].

Under hypothesis H_0, the characteristic function of z is

$$f_z^o(\xi) = f_y^o(\xi)^N = \left(B_1 + \frac{B}{1 - i\xi}\right)^N, \quad B = \exp(-b) \ , \quad B_1 = 1-B. \quad \text{(C-10)}$$

Upon expansion, there follows

$$f_z^o(\xi) = \sum_{n=0}^{N} \binom{N}{n} B_1^{N-n} \, B^n \, \frac{1}{(1 - i\xi)^n} \, . \qquad \text{(C-11)}$$

The corresponding probability density function of z under H_0 is

$$p_z^o(u) = B_1^N \, \delta(u) + \sum_{n=1}^{N} \binom{N}{n} B_1^{N-n} \, B^n \, \frac{u^{n-1} \exp(-u)}{(n-1)!} \quad \text{for } u \geq 0. \quad \text{(C-12)}$$

Finally, the exceedance distribution function of z under H_0, for threshold $v > 0$, is

$$\Pr(z > v) = \sum_{n=1}^{N} \binom{N}{n} B_1^{N-n} \, B^n \, Q_n(v) \ , \qquad \text{(C-13)}$$

where functions $\{Q_n(v)\}$ are defined in [7; page 12, set L = 0 in (23)].

Expansion (C-13) is a very useful one for evaluation of the exceedance distribution function, because it is a sum of positive terms, of which the components are quickly and easily evaluated by recursions. Thus, the false alarm probability P_f can be quickly and accurately determined. This procedure was utilized for the determination of figure 25.

APPENDIX D. PROCESSOR FORMS FOR VARIABLES WITH GAMMA DENSITY

In this appendix, we presume that all the bins are occupied when signal is present; that is, we have here, $\underline{M} = N$. Therefore, independent identically distributed random variables $\{x_n\}$, $1 \le n \le N$, have common probability density function

$$p(u) = \frac{\underline{a}^{\mu}\, u^{\mu-1} \exp(-\underline{a}u)}{\Gamma(\mu)} \quad \text{for } u > 0 \;, \tag{D-1}$$

regardless of the value of n, where

$$\underline{a} = \begin{Bmatrix} \underline{a}_1 < 1 & \text{for hypothesis } H_1 \\ \underline{a}_0 = 1 & \text{for hypothesis } H_0 \end{Bmatrix} \;; \quad \mu > 0 \;. \tag{D-2}$$

The characteristic function and cumulants of x_n are

$$f_x(\xi) = \frac{1}{(1 - i\xi/\underline{a})^{\mu}} \;, \qquad \chi_x(k) = (k-1)!\, \frac{\mu}{\underline{a}^k} \quad \text{for } k \ge 1 \;, \tag{D-3}$$

respectively. Thus, $\chi_x(1) = \mu/\underline{a}$, and, in particular, the mean of x_n is μ (not 1) for hypothesis H_0. All the previous results in [1; 2; 3] pertain to the special case of parameter $\mu = 1$ in (D-1). (If $\mu = I/2$, I integer, then random variable x_n in (D-1) and (D-3) is a scaled chi-squared variate with I degrees of freedom.)

GENERALIZED LIKELIHOOD RATIO PROCESSOR

The situation is identical to (D-1) and (D-2), except that strength parameter $\underline{a}_1$ is unknown. We hypothesize value a_1, where $a_1 \le 1$, yielding for the joint probability density function of observation $\{x_n\}$ under hypothesis H_1

$$p_1 \equiv p_1(x_1,\ldots,x_N) = \prod_{n=1}^{N} \left(\frac{a_1^{\mu}\, x_n^{\mu-1} \exp(-a_1 x_n)}{\Gamma(\mu)} \right) =$$

$$= \left(\frac{a_1^{\mu}}{\Gamma(\mu)} \right)^N \Pi^{\mu-1} \exp(-a_1 \Sigma), \qquad \text{(D-6)}$$

where random variables

$$\Pi \equiv \prod_{n=1}^{N} x_n , \quad \Sigma \equiv \sum_{n=1}^{N} x_n . \qquad \text{(D-7)}$$

In order to maximize p_1 in (D-6), we take a derivative with respect to a_1, obtaining

$$\frac{dp_1}{da_1} = \frac{\Pi^{\mu-1}}{\Gamma(\mu)^N} a_1^{\mu N-1} \exp(-a_1 \Sigma) \, (\mu N - a_1 \Sigma) . \qquad \text{(D-8)}$$

The best choice of a_1 is therefore the random variable

$$a_1 = \begin{Bmatrix} \frac{\mu N}{\Sigma} & \text{if } \Sigma > \mu N \\ 1 & \text{if } \Sigma \le \mu N \end{Bmatrix} , \qquad \text{(D-9)}$$

because $a_1 \le 1$ always. The resulting maximum of p_1 in (D-6) is because useful values of false alarm probability P_f will require threshold values v in (D-14) larger than μN, in practice. In fact, the mean of Σ is exactly μN under hypothesis H_0; therefore, v would have to be larger than the mean of Σ to realize small P_f.

The left-hand side of the processor in (D-14) does not need or utilize μ for its realization; however, achievement of a specified P_f would require knowledge of μ in order to set the correct threshold value for v.

This generalized likelihood ratio test is identical to the optimum processor test in (D-5). This result could have been anticipated, because the likelihood ratio test (D-5) was independent of μ and $\underline{a}_1$, meaning that $\underline{a}_1$ was never needed to realize the optimum processor. Again, the situation here is that the signal occupies all bins with equal powers when it is present. The following subsection treats the case of unknown and unequal signal powers per bin.

$$\frac{(\mu/e)^{\mu}}{\Gamma(\mu)\ x_n} \quad \text{for } n \ \varepsilon \ L \ , \tag{D-18}$$

and $x_n^{\mu-1} \exp(-x_n)/\Gamma(\mu)$ otherwise. Therefore,

$$\max\ p_1 = \prod_{n\varepsilon L}\left(\frac{(\mu/e)^{\mu}}{\Gamma(\mu)\ x_n}\right) \prod_{n\not\varepsilon L}\left(\frac{x_n^{\mu-1} \exp(-x_n)}{\Gamma(\mu)}\right) =$$

$$= \prod_{n\varepsilon L}\left(\frac{(\mu/e)^{\mu} \exp(x_n)}{x_n^{\mu}}\right) \prod_{n=1}^{N}\left(\frac{x_n^{\mu-1} \exp(-x_n)}{\Gamma(\mu)}\right) . \tag{D-19}$$

Meanwhile, the joint probability density function of observation $\{x_n\}$ under hypothesis H_0 is, from (D-15) and (D-7),

$$p_0 = \frac{\Pi^{\mu-1} \exp(-\Sigma)}{\Gamma(\mu)^N} . \tag{D-20}$$

The modified generalized likelihood ratio follows from (D-19) and (D-20) as (random variable)

$$\text{MGLR} = \frac{\max\ p_1}{p_0} = \prod_{n\varepsilon L}\frac{(\mu/e)^{\mu} \exp(x_n)}{x_n^{\mu}} = \prod_{n\varepsilon L}\exp\left(\mu\left[\frac{x_n}{\mu} - 1 - \ln\left(\frac{x_n}{\mu}\right)\right]\right) =$$

$$= \exp\left(\mu \sum_{n=1}^{N} g(x_n/\mu)\right) , \tag{D-21}$$

where nonlinearity g is independent of μ and is defined by

$$g(x) = \left\{\begin{matrix} x - 1 - \ln(x) & \text{for } x > b \\ 0 & \text{for } x \le b \end{matrix}\right\} . \tag{D-22}$$

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